1. Let $p \geq 5$ be prime. Let $G$ be a group of order $4p$. Show that:

[Compare this problem to HW9 problem 1, HW7 problem 3, Quiz 7+]

(a) The group $G$ contains a normal, cyclic $p$-sylow subgroup. What are the possible isomorphism types of a 2-sylow subgroup of $G$?

(b) If $G$ is abelian, then $G$ is isomorphic to either

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \text{ or } (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/p\mathbb{Z}.$$ 

(c) If $G$ is non-abelian and $p \equiv 3 \mod 4$, then $G$ is isomorphic to

$$D_{2p} \times \mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{Z}/p\mathbb{Z}, \rtimes \varphi \mathbb{Z}/4\mathbb{Z}$$

where $\varphi(1) = [-1] \in (\mathbb{Z}/p\mathbb{Z})^\times$. Deduce that up to isomorphism there are four groups of order $4p$ if $p \geq 5$ and $p \equiv 3 \mod 4$.

(d) Classify all groups $G'$ of order $12$ which contain a normal $3$-sylow subgroup. Show that if the $3$-sylow $P \leq G'$ of a group $G'$ of order $12$ is not normal, then the homomorphism $\phi : G' \to S_{G/P} \cong S_4$ defined by left multiplication is injective and $G' \cong A_4$. Deduce that up to isomorphism there are five groups of order $12 = 4 \cdot 3$.

2. Let $p$ and $q$ be distinct primes and $G$ be a group of order $p^2q$. Assume that $p \not\equiv \pm 1 \mod q$ and $q \not\equiv 1 \mod p$.

(a) Using the Sylow theorems, show that $n_p := |Syl_p(G)| = 1$ and $n_q := |Syl_q(G)| = 1$. Deduce that the (unique) $p$-sylow and $q$-sylow subgroups of $G$ are normal.

(b) Show $G$ is isomorphic to the product of its $p$-sylow and $q$-sylow subgroups. Classify (up to isomorphism) all groups of order $p^2q$ such that $p \not\equiv \pm 1 \mod q$ and $q \not\equiv 1 \mod p$.

(c) What is the smallest order $p^2q$ with the property that $p \not\equiv \pm 1 \mod q$ and $q \not\equiv 1 \mod p$?

3. In this exercise you’ll show that if $n \geq 5$, then the only normal subgroups of $A_n$ are $\{e\}$ and $A_n$. Recall that you have previously shown that the only normal subgroups of $S_n$ are $\{e\}$ and $A_n$ and $S_n$. This does not imply the result, as a subgroup $H \leq G$ may have more normal subgroups than those of the form $K \cap H$, where $K$ a normal subgroup of $G$. Here’s an example:

(a) Show that the normal subgroup $V \leq S_4$ contains a normal subgroup $H \leq V$ which is not normal in $S_4$. Conclude that the property of normality is not a transitive property. Show that $H$ is not of the form $K \cap V$ for some normal subgroup $K \leq S_4$. 


The next three parts show the only normal subgroups of $A_n$ are $\{e\}$ and $A_n$.

(b) Let $n$ be a positive integer and $K \leq A_n$ be a normal subgroup. Show that if $\sigma \in S_n$ is a 2-cycle then
   i. $K^\sigma := \sigma K \sigma^{-1}$ is a normal subgroup of $A_n$.
   ii. $K \cap K^\sigma$ is a normal subgroup of $S_n$.
   iii. $KK^\sigma$ is a normal subgroup of $S_n$.

(c) Let $n \geq 5$ and $K \leq A_n$ be a normal subgroup. Let $\sigma \in S_n$ be a 2-cycle. Show that if $K \neq A_n$, then $K \cap K^\sigma = \{e\}$. Show that if $K \neq \{e\}$, then $KK^\sigma = A_n$. Deduce that if $K$ is a non-trivial, proper subgroup of $A_n$, then $A_n \cong K \times K^\sigma \cong K^2$.

(d) Let $n \geq 5$ and $K \leq A_n$ be a non-trivial, proper normal subgroup. Show that the image of the diagonal $\Delta(K) := \{k(\sigma k \sigma^{-1}) : k \in K\} \leq KK^\sigma$ is a normal subgroup of $S_n$ isomorphic to $K$. Produce a contradiction and conclude:

**Theorem 0.1.** The only normal subgroups of $A_n$ are $\{e\}$ and $A_n$.

Groups $G$ with the property that the only normal subgroups of $G$ are $G$ and $\{e\}$ are called simple. This exercise showed $A_n$ is simple if $n \geq 5$. Another simple group is $\mathbb{Z}/p\mathbb{Z}$ (where $p$ is prime), as the only subgroups are $\mathbb{Z}/p\mathbb{Z}$ and $\{0\}$.

4. For each $n > 0$, let $P_n$ denote a (fixed) 2-sylow subgroup of $S_n$. Hence, $P_2 \cong \mathbb{Z}/2\mathbb{Z}$ and $P_4 \cong D_8, \ldots$

   (a) Show that $P_{2k+1}$ contains a subgroup isomorphic to $P_{2k} \times P_{2k}$ whose index in $P_{2k+1}$ is 2. Show that $P_{2k+1} \cong (P_{2k} \times P_{2k}) \rtimes \varphi \mathbb{Z}/2\mathbb{Z}$ where $\varphi(1)(x, y) = (y, x)$.

   (b) Show that if the binary expansion of $n$ is $2^k + a_{k-1}2^{k-1} + \ldots + a_0$, where $a_i \in \{0, 1\}$ then $P_n \cong \prod_{i=0}^{k}(P_2)^{a_i}$, where $(P_2)^0 := \{e\}$.

   (c) Show that if $n! = 2^m m$ where $\gcd(2, m) = 1$, then the number of 2-sylow subgroups of $S_n$ equals $m$. Deduce that $N_{S_n}(P_n) = P_n$. [Note that it is not generally the case that the normalizer of a $p$-sylow $P$ in $S_n$ is $P$. For example, in a previous homework you showed that the normalizer of a $p$-sylow in $S_p$ has order $p(p-1)$ and is isomorphic to $\mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^\times$. In general, the normalizer of a $p$-sylow $P$ in $S_n$ is $P$ if and only if $p=2$]
5. (Bonus, Ungraded) This problem determines the \( p \)-sylow subgroups of \( S_{p^2} \):

(a) Let \( p \) be a prime. What are the \( p \)-sylow subgroups of \( S_p \)? How many \( p \)-sylow subgroups are there? What is the order of a normalizer of a \( p \)-sylow subgroup?

(b) Show that the order of a \( p \)-sylow subgroup of \( S_{p^2} \) is \( p^{p+1} \).

(c) Let \( x \in S_{p^2} \) be a \( p^2 \)-cycle. What is the order of the conjugacy class of \( x \)? What is the order of the centralizer of \( x \)? Show that \( C_{S_{p^2}}(x) = \langle x \rangle \).

(d) Let \( P \leq S_{p^2} \) be a \( p \)-sylow subgroup containing the \( p^2 \)-cycle \( x \). Show that \( Z(P) \) is non-trivial and contained in \( \langle x \rangle \). What are the possible cycle types of elements in \( \langle x \rangle \)?

(e) Let \( y \in S_{p^2} \) be an element of \( p \)-power order. Show that \( y \) is contained in the center of some \( p \)-sylow subgroup of \( S_{p^2} \) if and only if \( p^{p+1} \) divides \( |C_{S_{p^2}}(y)| \). Deduce that \( Z(P) = \langle x^p \rangle \).

(f) Deduce that the center of each \( p \)-sylow subgroup of \( S_{p^2} \) is a cyclic subgroup generated by an element with cycle type \( \{p,p,p,\ldots,p\} \) (the product of \( p \) disjoint \( p \)-cycles), and that all subgroups generated by an element of this cycle type occur as the center of some \( p \)-sylow subgroup of \( S_{p^2} \). [Hint: use conjugation.]

(g) Let \( y_1, \ldots, y_p \in S_{p^2} \) be \( p \) pairwise disjoint \( p \)-cycles and \( y := \prod_{i=1}^p y_i \). Show that if \( P \) is a \( p \)-sylow subgroup \( S_{p^2} \) and \( \langle y \rangle = Z(P) \), then \( P \leq C_{S_{p^2}}(y) \). What is the size of the conjugacy class of \( y \)? What is the order of the centralizer of \( y \)? How many subgroups are there which are generated by an element with cycle type \( \{p,p,p,\ldots,p\} \)?

(h) Show that the centralizer \( C_{S_{p^2}}(y) \) acts on the set \( Y := \{y_1, y_2, \ldots, y_p\} \) by conjugation, i.e. for all \( y_j \in Y \) the conjugate \( g y_j g^{-1} = y_j \) for some \( y_j \in Y \). Prove that the associated homomorphism \( \phi : C_G(y) \to S_Y \cong S_p \) is surjective. Show that the kernel of \( \phi \) is \( \langle y_1 \rangle \langle y_2 \rangle \ldots \langle y_p \rangle \cong (\mathbb{Z}/p\mathbb{Z})^p \).

(i) Show there is a subgroup (there will be many) \( S \leq C_{S_{p^2}}(y) \) such that the restriction \( \phi : S \to S_Y \cong S_p \) is an isomorphism. Deduce that

\[
C_{S_{p^2}}(y) \cong \ker(\phi) S \cong (\mathbb{Z}/p\mathbb{Z})^p \rtimes \varphi S_p
\]

where \( \varphi(\sigma)(k_1, \ldots, k_p) = (k_{\sigma(1)}, \ldots, k_{\sigma(p)}) \). Conclude a \( p \)-sylow subgroup of \( S_{p^2} \) is isomorphic to

\[
C_{S_{p^2}}(y) \cong \ker(\phi) S \cong (\mathbb{Z}/p\mathbb{Z})^p \rtimes \varphi \mathbb{Z}/p\mathbb{Z}
\]

where \( \varphi(1)(k_1, k_2, \ldots, k_p) = (k_2, k_3, \ldots, k_{p-1}, k_1) \).

(j) Understanding check: If \( p = 2 \) then the \( p \)-sylow of \( S_8^2 \) is \( D_8 \). Find an explicit isomorphism \( D_8 \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \varphi \mathbb{Z}/2\mathbb{Z} \), where \( \varphi(1)(k_1, k_2) = (k_2, k_1) \).

(k) Show that the set of \( p \)-sylows \( P \leq S_{p^2} \) with center \( Z(P) = \langle y \rangle \) are in bijection with \( p \)-sylows of \( S_p \). Using this, determine the number of \( p \)-sylow subgroups of \( S_{p^2} \), and calculate the order of the normalizer of each (any) \( p \)-sylow subgroup.