Formula: Inverse of a 2x2 matrix.

Consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Flip on diagonal,

Negative ~~inverse~~ off diagonal

Compute

$$AB = \begin{bmatrix} ad - cb & 0 \\ 0 & -bc + ad \end{bmatrix}.$$

$$= \begin{bmatrix} ad - cb & 0 \\ 0 & ad - cb \end{bmatrix}.$$

$$= (ad - bc)\, I.$$

If $ad - bc \neq 0$ then

$$A \underbrace{\left(\frac{1}{ad - bc} B \right)}_{A^{-1}} = I$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad-bc \neq 0$.

If $ad-bc = 0$. Then

$AB = 0$ in otherwords

$$A\begin{bmatrix} d \\ -c \end{bmatrix} = 0 \quad \text{and}$$

$$A\begin{bmatrix} -b \\ a \end{bmatrix} = 0.$$

So either $A = 0$ matrix

or A has a non-zero vector in $\ker(A)$

$\Longrightarrow$ A is not invertible.

Fact: A 2×2 matrix A is invertible if and only if

$$ad - bc \neq 0,$$

in which case

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The quantity

$$ad - bc$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad +$$

$$-$$

is called the determinant.

It's non-zero when A is invertible.

☆ The determinant determines if A is invertible.

Examples:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$\det(A) = 1\cdot(-1) - 1\cdot 1$$
$$= -2.$$

$-2 \neq 0$ so A is invertible.

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$\det(A) = 1 \cdot 2 - 2 \cdot 1 = 0.$

So A is not invertible.

We could also see that A is not invertible 2nd row is a multiple of the 1st i.e. $\text{Rref}(A) \neq I$.