Also we have \( f(a_i) = 0 \) \((i=0, 1, \ldots, n-1) \Rightarrow a_0 \sim a_{n-1} \) are \( n \) distinct roots of the polynomial \( f(x) \). \( f(x) = k(x-a_0)(x-a_1) \ldots (x-a_{n-1}) \), plug in \( x = a_0 \), we have
\[
\det A = f(a_0) = (a_{n-1} - (x) - (a_{n-1} - a_0)) = n_{a_{n-1}}^{2} a_2 \]
the proof is completed.

For 32, we just plug in \( n = 4 \), \( a_0 = 1 \), \( a_4 = 5 \) into the formula.

we see the determinant \( = (1-4) (1-5) (1-6) (1-7) = (4-1)(4-2)(4-3)(4-4) = (3)(2)(1) \)
\( = 4! \cdot 3! \cdot 2! \cdot 1! = 24 \cdot 6 \cdot 2 = 288 \)

\( \vec{v} \) \( \sim \vec{v}_{n-1} \) linearly independent

43. Consider \( T = \operatorname{det} \left(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m, T \right) \) from \( R^n \rightarrow R \) determine its kernel & image.

\( \text{Ker}(T) = \{ \vec{x} \in R^n \mid T(\vec{x}) = 0 \} \)

\( \text{Im}(T) = \{ \lambda \vec{x} \mid \lambda \in R \} \)

TS = 0 if and only if the columns are linearly dependent. Since \( \vec{v}_1 \sim \vec{v}_{n-1} \) linearly independent,

\( \vec{x} \) is a linear combination of \( \vec{v}_1 \sim \vec{v}_{n-1} \). \( \vec{v}_1 \sim \vec{v}_{n-1} \) has \( \dim = n-1 \).

so we can choose \( \vec{x} \) s.t. \( \vec{x} \) is linearly independent with \( \vec{v}_1 \sim \vec{v}_{n-1} \) then \( TS = 0 \) by linearity of \( T \).

\( T(\vec{x}) = \lambda T(\vec{x}) = \vec{0} \), row dimension \( 1 \) \( \Rightarrow \text{Ker}(T) + \text{Im}(T) = n \), which confirms rank-nullity.

Video HW 44

6.2 34. a. Show that \( \text{ref}[A \mid AB] = [In]B \). A \( \text{In} \) is invertible.

Since \( A \) is invertible, \( \text{ref}[A^{-1}AB] = [In]M \) - we want to show \( M = B \).

Note \( [B]^{-1} \) has columns in the kernel of \( [A \mid AB] \), they are also in \( \text{ker}[In] \).

(row reduction doesn't change the kernel of a matrix). This means \( M = B \).

b. What does \( \text{ref}[A \mid AB] = [In]B \) tell you if \( B = A^{-1} \)?

In this case, the formula reads \( [A]^{-1} = [In]A^{-1} \). This is just the method we used to compute the inverse of a matrix \( A \).

66. a. Find a formula expressing \( d^n \) in terms of \( d_{n-1} \) & \( d_{n-2} \) \( n \geq 3 \).

expand \( d^n \) with respect to the first row, we'll get
\[ d_n = \det \begin{bmatrix} 1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ \end{bmatrix} - \det \begin{bmatrix} \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \end{bmatrix} \]

you can check that the first matrix is just \( d_{n-1} \) and \( \det \) of the second one is the minor box \( d_{n-2} \). Hence \( d_n = d_{n-1} - d_{n-2} \), \( d_1 = 1, d_2 = 0 \).

b. \( d_3 = 1, d_4 = 0, d_5 = 1, d_6 = 0, d_7 = 1, d_8 = 0 \). 

c. from the result above, see \( d_{n+3} = d_n \), \( d_{n+1} = d_n \).

d. \( d_{\infty} = d_4 = -1 \) from part (c).

70. a. Find the Fibonacci numbers \( f_0, f_1, \ldots, f_8 \):

\( f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21 \).

b. \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), prove by induction \( A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \).

\[ A^{n+1} = A^n \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix} \]

\( c. \ f_{n+1} f_{n-1} - f_n^2 = \det(A^n) = (\det A)^n = (-1)^n \) (since \( \det A = -1 \)).

63. 18. a. Sketch the ellipse when \( A = \begin{bmatrix} p & 0 \\ 0 & g \end{bmatrix} \), where \( p, g > 0 \). What's it's area?

[Diagram of an ellipse with semi-major and semi-minor axes labeled]

Area of \( T(\mathbb{S}^2) = \text{area of } \mathbb{S}^2 \cdot \det A = \pi \cdot pg = pg\pi \),

with semi-major & semi-minor axes \( p \& g \).

b. by part (a), we know area of \( T(\mathbb{S}^2) = \pi ab \); on the other hand, it is \( \pi \det(A) \).

\( \Rightarrow \) \( \det A = ab \)
follow the hint. Note \( T([-1]) = 4 [-1] \)
\& \( T([-1]) = 2 [-1] \)
we know the long axis of \( T(x) \) is the line \( y=x \) and \( a=4 \); the short axis of \( T(xy) \)
is the line \( y=-x \) and \( b=2 \).