5.4: #20) Find the least squares solution $\vec{x}^*$ to

$$A\vec{x} = \vec{b}, \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$  

Verify that $\vec{b} - A\vec{x}^*$ is indeed perpendicular to the image of $A$.

**Solution:** The least squares solutions to $A\vec{x} = \vec{b}$ are exactly the solutions to the normal equation $A^TA\vec{x}^* = A^T\vec{b}$. We compute the normal equation:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \implies \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

Solving the normal equation is straightforward, either by row reducing an augmented matrix, or by using an inverse matrix. Regardless, we obtain $\vec{x}^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Next, we note that

$$\vec{v} = \vec{b} - A\vec{x}^* = \begin{pmatrix} 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$  

In order to check that $\vec{v}$ is orthogonal to $\text{Im}(A)$, we can equivalently check that $\vec{v}$ is orthogonal to $\text{Span}\left( (1, 1, 0), (1, 0, 1) \right)$. Because of how spans work, to verify this we only need to check that $\vec{v}$ is orthogonal to each of the two spanning vectors. However, it is easy to see that $(-1, 1, 1) \cdot (1, 1, 0) = 0$, $(-1, 1, 1) \cdot (1, 0, 1) = 0$, and so we are done.
#22) Find the least squares solution \( \vec{x}^* \) to

\[
A \vec{x} = \vec{b}, \quad \text{where} \quad A = \begin{pmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}.
\]

Compute the error \( ||\vec{b} - A\vec{x}^*|| \).

**Solution:** Find the normal equation:

\[
\begin{pmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 50 & 41 \\ 41 & 38 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 68 \\ 47 \end{pmatrix}
\]

Solving the normal equation yields

\[
\vec{x}^* = \begin{pmatrix} 3 \\ -2 \end{pmatrix}
\]

We have an error of

\[
||\vec{b} - A\vec{x}^*|| = \left| \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right| = \left| \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right| = 0
\]

What just happened? Well, if you look closely, \( \vec{b} \) is equal to exactly three times the first column of \( A \) minus twice the second column of \( A \). So, by a stroke of good fortune, \( \vec{b} \) is in the image of \( A \). Because this system has solutions, of course the least squared error solution will just be the actual solution!
Find the least squares solution \( \mathbf{x}^* \) to

\[
A\mathbf{x} = \mathbf{b}, 
\text{ where } A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}.
\]

Draw a sketch showing \( \mathbf{b} \), \( \text{Im}(A) \), \( A\mathbf{x}^* \), \( \mathbf{b} - A\mathbf{x}^* \).

**Solution:** One more time! The normal equation is

\[
14x^* = 28
\]

I have opted to omit the vector symbol on \( x^* \) because in this problem \( x \) and \( x^* \) are vectors of length 1, better known as numbers. Regardless, it is clear that we get the solution \( x^* = \frac{2}{7} \).

Below is a picture of all of the desired elements. The x and y axis are labeled, but z is not due to clutter. The origin is the point the vectors are emanating from. In particular, the bottom corner of the cube is (0,-2,0), and not (0,0,0) as one might suspect.

The long blue line is \( \text{Span}(1,2,3) \), aka the image of \( A \). The green vector is \( \mathbf{b} = (3, 2, 7) \). The vector on the blue line is \( A\mathbf{x}^* = (2, 4, 6) \), the closest vector in \( \text{Im}(A) \) to \( \mathbf{b} \). The short red vector \((1,-2,1)\) is the error. As you can see, the error is indeed perpendicular to \( \text{Im}(A) \).