# Review Problems for the Second Midterm

The problems below are selected from the textbook's sections 3.4, 5.1–5.4, 6.2–6.3, and 7.1–7.3.

**1.** Is the vector $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ in the span of the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$?

**2.** Find constants $c_1, c_2, c_3$ so that

$$\begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

and describe your findings in terms of coordinates.

**3.** Consider the linear transformation $T(\vec{x}) = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \vec{x}$. Find the matrix of this linear transformation with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

**4.** Find an invertible $2 \times 2$ matrix $S$ such that

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} S$$

is of the form $\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$, and find the values of $b$ and $d$.

**5.** Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^4$. Can you find a vector $\vec{u}_4$ such that the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ are orthonormal? If you can, how many different choices for $\vec{u}_4$ are there?

**6.** Find constants $a, b, c, d, e, f, g$ so that the three vectors

$$\begin{bmatrix} a \\ d \\ f \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ g \end{bmatrix}, \begin{bmatrix} c \\ e \\ 1/2 \end{bmatrix}$$

are orthonormal.

**7.** Find the orthogonal projection of $9\vec{e}_1$ onto the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

**8.** If $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5$ are orthonormal vectors in $\mathbb{R}^{10}$, find the length of the vector

$$\vec{x} = 7\vec{u}_1 - 3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 - \vec{u}_5.$$

**9.** Find an orthonormal basis for the image of $A$, where $A$ is the matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

**10.** Find an orthonormal basis of the plane in $\mathbb{R}^3$

$$x_1 + x_2 + x_3 = 0.$$

**11.** Find an orthonormal basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ of $\mathbb{R}^3$ such that

$$\text{span}(\vec{u}_1) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

and

$$\text{span}(\vec{u}_1, \vec{u}_2) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right).$$

**12.** If the $n \times n$ matrices $A$ and $B$ are orthogonal, which of the following matrices must be orthogonal, and which may not be orthogonal?

(a) $3A$

(b) $-B$

(c) $AB$

(d) $A + B$

**13.** Find an orthogonal transformation $T$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

**14.** Let $V$ be the solution space of the linear system:

$$x_1 + x_2 + x_3 + x_4 = 0$$

$$x_1 + 2x_2 + 5x_3 + 4x_4 = 0$$

and find a basis of $V^\perp$.

**15.** Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$, when $A = \begin{bmatrix} 1 & 1 \\ 2 & 8 \\ 1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. Determine the error $\|A\vec{x}^* - \vec{b}\|$, and explain your answers.

**16.** Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(0,3), (1,3), (1,6)$ using least squares, and sketch your solution along with the data points.

**17.** Find the determinant of the matrix $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{bmatrix}$.

**18.** If $A$ is a $4 \times 4$ matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$, and if $\det(A) = 8$, find following determinants:

(a) $\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ -9\vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$

(b) $\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 + 9\vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$

(c) $\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_1 + \vec{v}_2 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \end{bmatrix}$

(d) $\det \begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix}$

**19.** Find the derivative of the function

$$f(x) = \det \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

**20.** Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose that for some $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, we know that $T(\vec{v}_1) = 3\vec{v}_1$ and that $T(\vec{v}_2) = 4\vec{v}_2$. What can you say about $\det(A)$?

**21.** Find the area of the parallelogram defined by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

**22.** If $A$ is an invertible $n \times n$ matrix, and $\vec{v}$ is an eigenvector of $A$ with associated eigenvalue $\lambda$:

(a) Is $\vec{v}$ an eigenvector of $A^3$? If so, what is the eigenvalue?

(b) Is $\vec{v}$ an eigenvector of $A + 2I_n$? If so, what is the eigenvalue?

**23.** Find all $2 \times 2$ matrices for which $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with associated eigenvalue $-1$.

**24.** If $\vec{v}$ is an eigenvector of the matrix $A$, show that $\vec{v}$ is either in the image of $A$ or the kernel of $A$.

**25.** Find an eigenbasis for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, and then diagonalize $A$.

**26.** Find all real eigenvalues of the matrix $\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$, and their algebraic multiplicities.

**27.** Consider the matrix $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$, where $k$ is some constant. For which values of $k$ does $A$ have two distinct real eigenvalues? When are there no real eigenvalues?

**28.** For each of the following matrices, find all (real) eigenvalues. Then find a basis of each eigenspace, and diagonalize $A$ if possible:

(a) $A = \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$

**29.** Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Is there an eigenbasis?

**30.** Consider a subspace $V$ of $\mathbb{R}^n$ with $\dim(V) = m$.

(a) Let the $n \times n$ matrix $A$ represent orthogonal projection onto $V$. What can you say about the eigenvalues of $A$ and their algebraic and geometric multiplicities?

(b) Let the $n \times n$ matrix $B$ represent reflection about $V$. What can you say about the eigenvalues of $B$ and their algebraic and geometric multiplicities?