Review: Matrix Powers

+ long term limits.

Problem: Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}. \text{ Find}$$

A closed form for A^t

for all powers t.

IDEA: MATRIX POWERS w/

DIAGONAL MATRICES

ARE EASY.

DIAGONALIZE A!

$$A^t = (SDS^{-1})^t = SD^tS^{-1}.$$

DIAGONALIZE A.

① FIND EIGENVALUES.

$$p_A(t) = \begin{vmatrix} 1-t & 2 \\ 3 & 6-t \end{vmatrix}$$

$$= (1-t)(6-t) - 6$$

$$= t^2 - 7t$$

$$\lambda = 0, 7.$$

② FIND EIGENVECTORS

$\lambda = 7$

$$E_7 = \operatorname{Ker}(A - 7I) = \operatorname{Ker}\left(\begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix}\right)$$

$$\begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} 0 & 0 \\ 3 & -1 \end{pmatrix}$$

$$\text{Kernel} = \operatorname{Span}\left(\begin{pmatrix} +1 \\ 3 \end{pmatrix}\right)$$

$$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{e.v. } \lambda = 7.$$

$\lambda = 0$

$$E_0 = \operatorname{Ker}(A) = \operatorname{Ker}\left(\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \xrightarrow{\text{r.r.}} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\text{Kernel} = \operatorname{Span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right).$$

$$v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{e.v.} \quad \lambda = 0.$$

③ Put it all together.

$$S = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix} \frac{1}{-7}$$

$$\begin{aligned} A &= SDS^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix} \frac{1}{-7}. \end{aligned}$$

$$A^t = S D^t S^{-1}$$

$$\Big(\text{why? } A \cdot A \cdot \ldots \cdot A =$$

$$S D \cancel{S^{-1} S} D \cancel{S^{-1} S} D S^{-1} \ldots$$

$$= S D \cdots D \cdot S^{-1} \Big)$$

$$A^t = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 7^t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix} \frac{1}{-7}$$

$$= \begin{pmatrix} 7^t & 0 \\ 3 \cdot 7^t & 0 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix} \frac{1}{-7}$$

$$= \begin{pmatrix} 7^{t-1} & 2 \cdot 7^{t-1} \\ 3 \cdot 7^{t-1} & 6 \cdot 7^{t-1} \end{pmatrix}.$$

Problem: Let $A = \begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix}$

$x_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$. Find a formula

for $A^t x_0$. Determine $\lim_{t \to \infty} A^t x_0$.

Idea: Matrix powers w/ diagonal matrices are easy.

Diagonalize A!

1 $p_A(t) = \begin{vmatrix} 0.3-t & 1 \\ 0.7 & -t \end{vmatrix}$

$$= (0.3-t)(-t) - 0.7$$

$$t^2 - 0.3t - 0.7.$$

$$\lambda = \frac{0.3 \pm \sqrt{0.09 + 2.8}}{2}$$

$$= \frac{0.3 \pm \sqrt{\frac{289}{100}}}{2}.$$

Compute

$$\sqrt{289}$$

to do this factorize 289.

p smallest prime factor

$$\cancel{2}, \cancel{3}, \cancel{5}, \cancel{7}, \cancel{11}, \cancel{13}, 17$$

$$17^2 = 289.$$

$$\lambda = \frac{1}{2}(0.3 \pm 1.7) = 1, -0.7.$$

$$A = \begin{pmatrix} 0.3 & 1 \\ 0.7 & 0 \end{pmatrix}$$

② Find eigenvectors

$$E_{-0.7} = \ker(A + 0.7I)$$

$$= \ker\left(\begin{pmatrix} 1 & 1 \\ .7 & .7 \end{pmatrix}\right)$$

$$\begin{pmatrix} 1 & 1 \\ .7 & .7 \end{pmatrix} \xrightarrow{r.r} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{kernel} = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{e.v. } \lambda = -0.7.$$

$$E_1 = \ker(A - I)$$
$$= \ker \begin{pmatrix} -.7 & 1 \\ .7 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -.7 & 1 \\ 0.7 & -1 \end{pmatrix} \xrightarrow{r.r} \begin{pmatrix} -0.7 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{Kernel} = \text{Span}\left(\begin{bmatrix} 1 \\ 0.7 \end{bmatrix}\right).$$
$$= \text{Span}\left(\begin{bmatrix} 10 \\ 7 \end{bmatrix}\right).$$

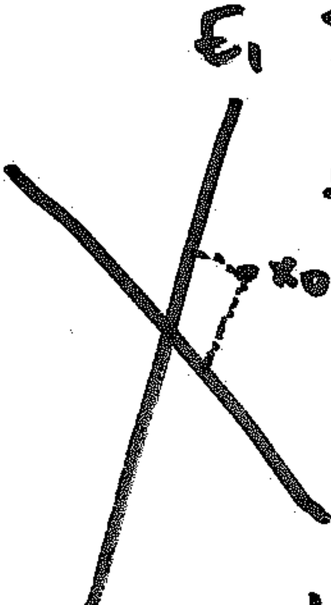
$$v_2 = \begin{bmatrix} 10 \\ 7 \end{bmatrix} \text{ e.v. } \lambda = 1.$$

③ Write $A = SDS^{-1}$

Compute $A^t x_0 = SD^t S^{-1} x_0$

$$x_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

Alternatively:



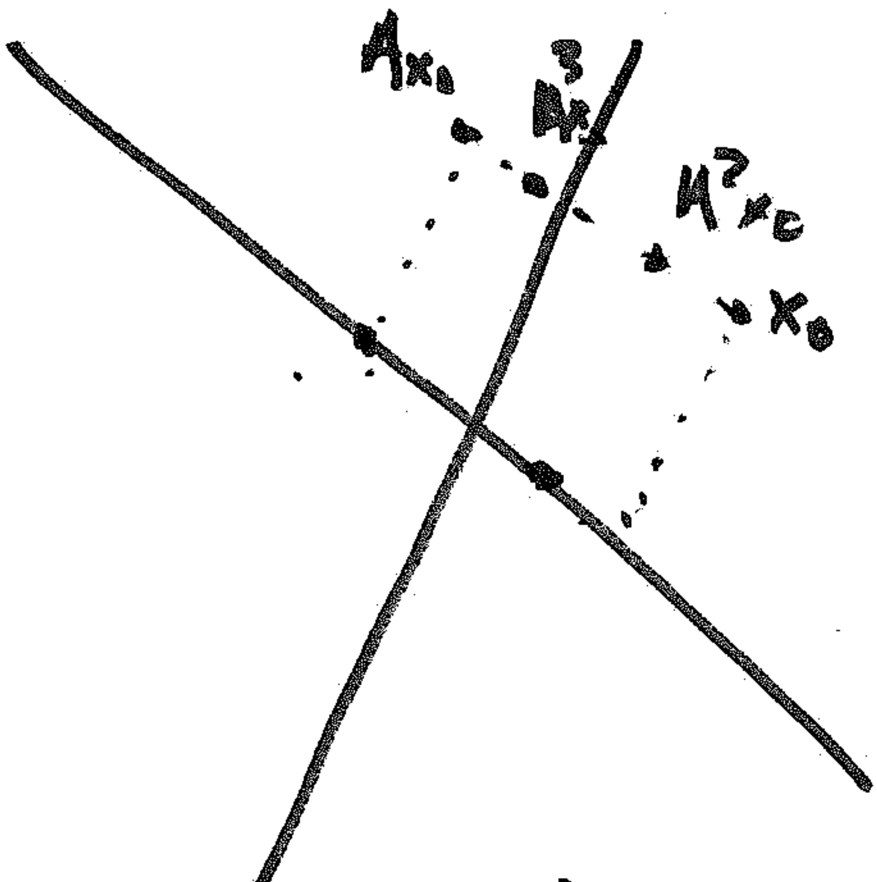
Since v_1, v_2 are a basis for $\mathbb{R}^2$, we can write

$$x_0 = c_1 v_1 + c_2 v_2$$

where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} x_0$.

Then

$$A^t x_0 = c_1 A^t x_0 + c_2 A^t v_2$$
$$= c_1 (-0.7)^t v_1 + c_2 1^t v_2$$
$$= c_1 (-0.7)^t v_1 + c_2 v_2$$



as $t \to \infty$ $A^t x_0 = c_2 v_2$.

Last step: Find S and S^{-1} and compute $S^{-1}x_0$.

$$S = \begin{array}{c} \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix} \end{array}$$

$$S^{-1} = \begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix} \frac{1}{17}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1}x_0 = S^{-1}\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7 \cdot 0.4 - 6}{17} \\ 1/17 \end{bmatrix} = \begin{bmatrix} \frac{2.8-6}{17} \\ 1/17 \end{bmatrix}.$$

$$= \begin{bmatrix} -3.2/17 \\ 1/17 \end{bmatrix}.$$

Formula:

$$A^t x_0 = c_1 (-0.7)^t v_1 + c_2 v_2$$

$$= (-0.7)^t \frac{(-3.2)}{17} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{17} \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

Limit

$$\lim_{t \to \infty} A^t x_0 = \frac{1}{17} \begin{bmatrix} 10 \\ 7 \end{bmatrix}.$$

Problem: Let $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$.

Does $\lim\limits_{t \to \infty} A^t e_1$ exist?

Solution: If A diagonalizable

$$A^t = S D^t S^{-1}$$

To analyze A^t find / analyze eigenvalues.

$$P_A(t) = -t(-1-t) + 1$$
$$= t^2 + t + 1.$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$
$$= \frac{-1 \pm \sqrt{3}\, i}{2}.$$

Complex eigenvalues $\Longrightarrow$

$A = SBS^{-1}$ where

$$B = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Where

$$M = r(\cos\theta + i\sin\theta)$$

what is r

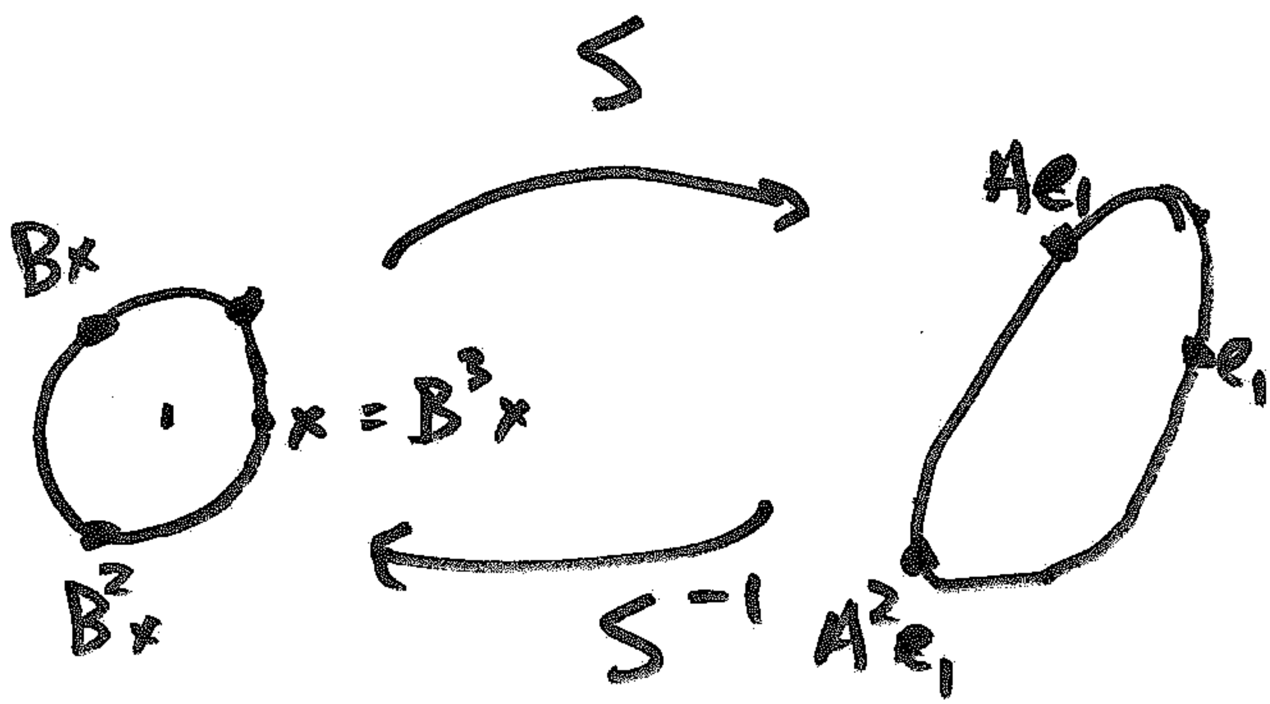
$$|M| = r = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 1.$$

B is a rotation matrix

$$M = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

rotation by 120°.



$\lim_{n\to\infty} B^n x$ does not exist

(vectors orbit around 0 along circles)

$\lim_{n\to\infty} A^n e_1$ does not exist.

(vectors orbit around 0 along ellipses)

$\cancel{\lim_{n\to\infty}}$ (if $r<1$ limit does exist

$r \geq 1$ limit does not exist)