Name: Solutions

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MATH 201, QUIZ 9

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Observe that $A - I$ has rank 2. Diagonalize A over the complex numbers, and determine if there a positive integer $t > 0$ such that $A^t = I$?

$$0 = \det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(-\lambda(1-\lambda) - 1)$$

$$= -(\lambda - 1)(\lambda^2 - \lambda - 1)$$

3 distinct eigenvalues, so A is diagonalizable. $\longrightarrow$ $\lambda = 1 \qquad \lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Thus, A is similar to the matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

; there is some matrix S s.t. $A = SDS^{-1}$

& $A^t = I_n \iff SD^tS^{-1} = I_n$

$$D^t = S^{-1}S = I_n$$

$$D^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^t & 0 \\ 0 & 0 & \left(\frac{1-\sqrt{5}}{2}\right)^t \end{bmatrix}$$

but $\frac{1+\sqrt{5}}{2} > 1$, so $\left(\frac{1+\sqrt{5}}{2}\right)^t > 1$ for all $t > 0$.

$\boxed{\text{Thus, there is \underline{no} positive integer } t \text{ so that } A^t = I_n.}$

We can find S by finding an eigenbasis for A:

$\lambda = 1$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_2 = 0 \\ x_3 = 0 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = \frac{1+\sqrt{5}}{2}$

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 0 & 2 \\ 0 & \frac{-1-\sqrt{5}}{2} & 1 \\ 0 & 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 0 & 2 \\ 0 & 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \frac{1-\sqrt{5}}{2} x_1 = -2x_3 \\ x_2 = \frac{1-\sqrt{5}}{2} x_3 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} \frac{-4}{1-\sqrt{5}} \\ \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$\lambda = \frac{1-\sqrt{5}}{2}$

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 & 2 \\ 0 & \frac{-1+\sqrt{5}}{2} & 1 \\ 0 & 1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 & 2 \\ 0 & 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \frac{1+\sqrt{5}}{2} x_1 = -2x_3 \\ x_2 = \frac{1+\sqrt{5}}{2} x_3 \end{matrix} \quad \vec{v}_3 = \begin{bmatrix} \frac{-4}{1+\sqrt{5}} \\ \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

So we take $S = \begin{bmatrix} 1 & \frac{-4}{1-\sqrt{5}} & \frac{-4}{1+\sqrt{5}} \\ 0 & \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 0 & 1 & 1 \end{bmatrix}$.