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**MATH 201, QUIZ 8.**

(1) Express the determinant $\begin{vmatrix} x & 0 & 0 & 1 \\ y & 1 & 1 & 2 \\ z & 0 & 1 & 2 \\ w & 2 & 1 & 0 \end{vmatrix}$ as a linear function of the variables $x, y, z, w$.

When one column of a matrix is a bunch of letters, we expand in that column:

$$\det\begin{bmatrix} X & 0 & 0 & 1 \\ Y & 1 & 1 & 2 \\ Z & 0 & 1 & 2 \\ \omega & 2 & 1 & 0 \end{bmatrix} = x\det\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} - y\det\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$+ z\det\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} - \omega\det\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= x\big(1\cdot(1\cdot 0 - 2\cdot 1) - 1(0\cdot 0 - 2\cdot 2) + 2(0\cdot 1 - 1\cdot 2)\big)$$

$$- y\,(1\cdot(0\cdot 1 - 1\cdot 2)) + z\,(1\cdot(1\cdot 1 - 1\cdot 2))$$

$$- \omega\,(1\cdot(1\cdot 1 - 1\cdot 0))$$

$$= x(-2+4-4) - y(-2) + z(-1) - \omega(1)$$

$$\boxed{\det\begin{bmatrix} X & 0 & 0 & 1 \\ Y & 1 & 1 & 2 \\ Z & 0 & 1 & 2 \\ \omega & 2 & 1 & 0 \end{bmatrix} = -2x + 2y - z - \omega}$$

(2) The kernel of the linear function in part (1) is exactly the space spanned by the columns of the matrix:

$$\text{span}\left(\begin{bmatrix}0\\1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\\0\end{bmatrix}\right).$$

Can you explain this fact geometrically?

Consider the parallelipiped determined by $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\\2\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\2\\2\\0\end{bmatrix}$.

If $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ lies in the span of the other three vectors, this parallelipiped lies in a lower dimensional subspace of $\mathbb{R}^4$, and thus its volume is "crushed down to 0", so in this case $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ is in the kernel of the linear transformation from part (a).

On the other hand, if $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ is <u>not</u> in this span, then we have four linearly independent vectors, which will form a parallelipiped with nonzero volume, so in this case $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ does <u>not</u> belong to the indicated kernel.

Hint: you may use the fact in part (2) to double check your math in part (1).