This exam contains 17 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- If you use a “fundamental theorem” you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. Consider the system of linear equations:

\[
2x_1 + 8x_2 + 4x_3 = 2, \\
2x_1 + 5x_2 + x_3 = 5, \\
4x_1 + 10x_2 - x_3 = 1.
\]

(a) (5 points) Express this system as a single matrix equation \( Ax = b. \)

This system can be expressed as

\[
\begin{bmatrix}
2 & 8 & 4 \\
2 & 5 & 1 \\
4 & 10 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
2 \\
5 \\
1
\end{bmatrix}
\]

(b) (20 points) Is \( A \) invertible? If so, compute \( A^{-1}. \) Find \( \det(A). \)

Row reduce:

\[
\begin{bmatrix}
2 & 8 & 4 & 1 & 0 & 0 \\
2 & 5 & 1 & 0 & 1 & 0 \\
4 & 10 & -1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
1 & 4 & 2 & \frac{1}{2} & 0 & 0 \\
0 & -3 & -3 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{3}{2}
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
1 & 4 & 2 & \frac{1}{2} & \frac{8}{3} & \frac{2}{3} \\
0 & -3 & -3 & -1 & \frac{3}{2} & -\frac{3}{2} \\
0 & 0 & 1 & 0 & \frac{3}{2} & \frac{3}{2}
\end{bmatrix}
\]

so: \( A \) is invertible

\[
A^{-1} = \begin{bmatrix}
\frac{8}{3} & \frac{8}{3} & \frac{2}{3} \\
\frac{1}{3} & -1 & \frac{1}{3} \\
0 & \frac{2}{3} & \frac{1}{3}
\end{bmatrix}
\]

and in row-reducing \( A \) to \( I_3, \) we multiplied its determinant by \( \frac{1}{2}, \frac{1}{3}, \frac{1}{3} = 18, \) so \( \frac{1}{18} \det(A) = \det(I_3) = 1, \) so \( \det(A) = 18. \)
(c) (10 points) Find all solutions to the system of linear equations above.

Since \( A \) is invertible, there is one solution, and we compute it by calculating

\[
\mathbf{x} = A^{-1} \mathbf{b}
\]

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 
\end{bmatrix} = \begin{bmatrix}
  \frac{-5}{6} & \frac{8}{3} & \frac{2}{3} \\
  \frac{1}{3} & -1 & \frac{1}{3} \\
  0 & \frac{2}{3} & \frac{1}{3}
\end{bmatrix} \begin{bmatrix}
  2 \\
  5 \\
  1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  \frac{-10}{6} + \frac{80}{6} - \frac{4}{6} \\
  \frac{2}{3} - \frac{15}{6} + \frac{1}{3} \\
  \frac{10}{3} - \frac{1}{3}
\end{bmatrix} = \begin{bmatrix}
  \frac{66}{6} \\
  \frac{-12}{3} + \frac{4}{3} \\
  \frac{1}{3}
\end{bmatrix} = \begin{bmatrix}
  11 \\
  -4 \\
  3
\end{bmatrix}
\]

And we may check:
\[
2(11) + 8(-4) + 4(3) = 34 - 32 = 2 \quad \checkmark
\]
\[
2(11) + 5(-4) + 3 = 25 - 20 = 5 \quad \checkmark
\]
\[
4(11) + 10(-4) - 3 = 44 - 40 = 1 \quad \checkmark
\]
2. (a) (25 points) Let $V$ be the subspace of $\mathbb{R}^3$ consisting of all vectors $v$ whose coordinate entries sum to 0. Find an orthonormal basis for $V$.

$$V = \{ \vec{x} : x_1 + x_2 + x_3 = 0 \}$$

$$= \ker \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad x_2, x_3 \text{ are free}$$

$$x_1 = -x_2 - x_3$$

$$V = \left\{ \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \text{ are real} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{these 2 vectors are a basis for } V.$$ 

We apply Gram-Schmidt:

$$\|v_1\| = \sqrt{1^2 + 1^2} = \sqrt{2}, \text{ so } u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_1 \cdot u_2 = \frac{1}{\sqrt{2}} (1 + 0 + 0) = \frac{1}{\sqrt{2}}$$

$$v_2^\perp = v_2 - (u_1 \cdot v_2) u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

$$\|v_2^\perp\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1^2} = \sqrt{\frac{3}{2}}, \text{ so } u_2 = \frac{\sqrt{2}}{3} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

So an orthonormal basis for $V$ is

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{\sqrt{2}}{3} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} \right\}.$$
(b) (3 points) Let $v_1, \ldots, v_k$ be the basis for $V$ which you found in part (a). How many orthogonal matrices $A$ are there with the property that the first $k$ columns of $A$ are $v_1, \ldots, v_k$, respectively? Reason geometrically. You are not required to find these matrices explicitly.

The columns of an orthogonal matrix are an orthonormal basis of $\mathbb{R}^n$ (here, $n=3$, so we only are looking at one more column after the first two).

Thus, the third column of $A$, considered as a vector, must lie in $V^\perp$, which is a line, and must be a unit vector.

There are only two such vectors, since $V^\perp$ intersects the unit sphere at two points.

Therefore, there are exactly two such matrices $A$.

A very, very rough sketch is given to the left.
3. (25 points) Consider the data set:

<table>
<thead>
<tr>
<th>Input ($i$)</th>
<th>Experimental Output ($y_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Find the quadratic polynomial $f(x)$ of the form $f(x) = ax^2 + b$ for which the quantity

$$
\epsilon := \left\| \sum_{i=0}^{2}(f(i) - y_i)^2 \right\|
$$

is minimized. Graph the best fit curve $f(x)$ and the data set together on a coordinate plane.

We're trying to solve

$$
\begin{bmatrix}
  f(0) \\
  f(1) \\
  f(2)
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  1 \\
  2
\end{bmatrix}
$$

in the least-squares sense.

$$
\begin{bmatrix}
  a \cdot 0^2 + b \\
  a \cdot 1^2 + b \\
  a \cdot 2^2 + b
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  1 \\
  2
\end{bmatrix}
$$

$$
\begin{bmatrix}
  0 & 1 & 4 \\
  1 & 1 & 1 \\
  4 & 1 & 1
\end{bmatrix} \begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix} = \begin{bmatrix}
  0 \\
  1 \\
  2
\end{bmatrix}
$$

We recall that we do this by solving $A^T A x^* = A^T \tilde{z}$, which we do on the next page.
\[ A^T = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}, \quad A^+ A = \begin{bmatrix} 17 & 5 \\ 5 & 3 \end{bmatrix} \]

\[ A^T \hat{z} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 17 & 5 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} \]

Row-reduce
\[ \begin{bmatrix} 17 & 5 & 9 \\ 5 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & -4 & 0 \\ 5 & 3 & 3 \end{bmatrix} \cdot \frac{1}{2} \to \begin{bmatrix} 1 & -2 & 0 \\ 5 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 13 & 3 \end{bmatrix} \]
\[ = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & \frac{3}{13} \end{bmatrix} \]

\[ a = \frac{6}{13}, \quad b = \frac{3}{13} \]

Best-fit curve is:
\[ F(x) = \frac{6}{13} x^2 + \frac{3}{13} \]

2 + Data

\[ F(0) = \frac{3}{13}, \quad F(1) = \frac{4}{13}, \quad F(2) = \frac{22}{13} \]

\[ F(x), \text{ the best-fit curve of the form } ax^2 + b. \]
4. Let \( B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \).

(a) (22 points) Find a basis for \( \mathbb{R}^2 \) consisting of eigenvectors for \( B \).

\[
\text{Find eigenvalues:} \quad \det (B - \lambda I_2) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 + \lambda - 1
\]

\[
\lambda = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}
\]

Eigenvecotr of \( \frac{-1+\sqrt{5}}{2} \):

\[
\begin{bmatrix} 1+\sqrt{5} \\ -1+\sqrt{5} \end{bmatrix}
\]

Row reduce:

\[
\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

Set \( b = 1 \):

\[
\vec{v}_1 = \begin{bmatrix} 1+\sqrt{5} \\ 1 \end{bmatrix} \quad \text{eigenvector of } B \quad \text{with eigenvalue } \frac{-1+\sqrt{5}}{2}
\]

Eigenvecotr of \( \frac{-1-\sqrt{5}}{2} \):

\[
\begin{bmatrix} 1-\sqrt{5} \\ -1-\sqrt{5} \end{bmatrix}
\]

Row reduce:

\[
\begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{c + d = 0}
\]

Set \( d = 1 \):

\[
\vec{v}_2 = \begin{bmatrix} 1-\sqrt{5} \\ 1 \end{bmatrix} \quad \text{eigenvector of } B \quad \text{with eigenvalue } \frac{-1-\sqrt{5}}{2}
\]

Thus, a basis of \( \mathbb{R}^2 \) consisting of eigenvectors of \( B \) is \( \{ \begin{bmatrix} 1+\sqrt{5} \\ 1 \end{bmatrix}, \begin{bmatrix} 1-\sqrt{5} \\ 1 \end{bmatrix} \} \).
(b) (11 points) Find a closed form expression for $B^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for all integers $t$.

For $S = \begin{bmatrix} \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$, then $S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1 + \sqrt{5}}{2} \\ -1 & \frac{1 + \sqrt{5}}{2} \end{bmatrix}$.

We use $S^{-1}$ to write $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in $u, v$ coordinates:

$S^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1 + \sqrt{5}}{2} \\ -1 & \frac{1 + \sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{5}} \begin{bmatrix} 2 + \frac{1 + \sqrt{5}}{2} \\ -2 + \frac{1 + \sqrt{5}}{2} \end{bmatrix}$

So that $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \left( (2 + \frac{1 + \sqrt{5}}{2}) u_1 + (-2 + \frac{1 + \sqrt{5}}{2}) v_2 \right)$.

Then, since $u_1, v_2$ are eigenvectors of $B$:

$B^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \left( (2 + \frac{1 + \sqrt{5}}{2}) B^t u_1 + (-2 + \frac{1 + \sqrt{5}}{2}) B^t v_2 \right)$

$= \frac{1}{\sqrt{5}} \left( (2 + \frac{1 + \sqrt{5}}{2}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 + \sqrt{5} \\ 2 \end{bmatrix} \right) + \left( -2 + \frac{1 + \sqrt{5}}{2} \right) \left( \begin{bmatrix} 1 - \sqrt{5} \\ 2 \end{bmatrix} \begin{bmatrix} 1 + \sqrt{5} \\ 2 \end{bmatrix} \right)$

is the desired closed-form expression.
(c) (6 points) Find a positive integer \( t \) such that \( \| B^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \| > 10^9 \). Justify your answer. You do not need to find the smallest such \( t \).

We note that \( \left| \frac{\sqrt{5} - 1}{2} \right| < \frac{3}{4} \), so that \( (-1 + \sqrt{5})^t \rightarrow 0 \), as \( t \rightarrow \infty \), \( (-\frac{1 + \sqrt{5}}{2}) \rightarrow \frac{1}{2} \).

On the other hand, \( \left| \frac{-1 - \sqrt{5}}{2} \right| > \frac{3}{2} \), and thus \( \left| (-1 - \sqrt{5})^2 \right| > 2 \).

We also notice that \( \|v_1\| \leq 10 \), and \( \|v_2\| \leq 1 \), and that

\[
\left| (2 + \frac{-1 + \sqrt{5}}{2}) \right| \leq 5
\]

\[
\left| (2 + \frac{-1 - \sqrt{5}}{2}) \right| \geq \left| 2 - \frac{(1 + \sqrt{5})}{2} \right|
\]

\[
\geq \frac{1}{4}.
\]

We happen to know that \( 2^4 = 16 > 10^1 \). So, consider \( t = 1200 \). Then,

\[
\| B^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \| = \| \left( 2 + \frac{-1 + \sqrt{5}}{2} \right)(\frac{-1 + \sqrt{5}}{2})^t v_1 + (2 + \frac{1 + \sqrt{5}}{2})(\frac{-1 - \sqrt{5}}{2})^t v_2 \|
\]

Picture:

- Stuff with \( v_2 \)
- \( B^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)
- Stuff with
4. (c) continued

So, by how triangles work,

$$\|B^t E_2 \| \geq \| ( - 2 + \frac{1 + \sqrt{5}}{2} ) ( \frac{1 - \sqrt{5}}{2} ) v_2 \| - \| ( 2 + \frac{1 + \sqrt{5}}{2} ) ( \frac{1 - \sqrt{5}}{2} ) v_1 \|$$

And we use our previous information:

$$\| B^{1200} E_2 \| \geq \frac{1}{4} \cdot 2^{600} \cdot 1 - 5 \cdot (\frac{1}{2})^{400} \cdot 10$$

$$= \frac{16}{4} \cdot 2^{400} \cdot 1 - \frac{50}{64} (\frac{1}{2})^{304} \quad (2^6 = 64)$$

$$\geq 10^{144} - 1 > 10^9$$

Thus, for \( t = 1200 \),

$$\| B^t E_2 \| > 10^9.$$
5. (20 points) Let

\[
A = \begin{bmatrix}
6 & 4 & 2 \\
7 & 3 & 4 \\
7 & 1 & 6 \\
2 & 1 & 1 \\
1 & 1 & 0 \\
3 & 1 & 2
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 4 & 6 & 7 & 1 & 1
\end{bmatrix}
\]

Determine the rank of A.

A goes from \( \mathbb{R}^7 \) to \( \mathbb{R}^6 \); by this picture

\[
\mathbb{R}^7 \xrightarrow{C} \mathbb{R}^3 \xrightarrow{B} \mathbb{R}^6
\]

we see that

\[\text{im}(A) \text{ is inside of im}(B); \text{ if } \vec{x} = A\vec{y}, \text{ then } \vec{x} = B(C\vec{y}) \text{ as well.}\]

What is im(B)? Row-reduce B to find out:

\[
\begin{array}{cccccc}
6 & 4 & 2 & \text{-6} & \text{-7} & \text{-7} \\
7 & 3 & 4 & \text{-7} & \text{-7} & \text{-7} \\
7 & 1 & 6 & \text{-7} & \text{-7} & \text{-7} \\
2 & 1 & 1 & \text{-7} & \text{-7} & \text{-7} \\
1 & 1 & 0 & \text{-7} & \text{-7} & \text{-7} \\
3 & 1 & 2 & \text{-7} & \text{-7} & \text{-7}
\end{array}
\]

so \( \text{im}(B) \) is the span of the first two columns of \( B \). It is two-dimensional.
Now, the image of $A$ might not be all of $\text{im}(B)$; it's the part of $\text{im}(B)$ that we can get to from $\text{im}(C)$.

What is $\text{im}(C)$? Row-reduce $C$ to find out:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 4 & 6 & 7 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & -1 & -3 & -12 & -15 & -18
\end{bmatrix}
\]

and we see that the image of $C$ has three dimensions, and is therefore all of $\mathbb{R}^3$.

Since the image of $C$ is all of $\mathbb{R}^3$, if we have an $\vec{x}$ in the image of $B$:

$\vec{x} = B\vec{y}$ and we know $\vec{y} = C\vec{z}$, since $\vec{y}$ is in $\mathbb{R}^3$ and is thus in the image of $C$.

Thus, $\vec{x} = BC\vec{z} = A\vec{z}$, so the image of $B$ is also inside of the image of $A$. 
Therefore, the image of $A$ is the image of $B$, so the image of $A$ is two-dimensional, so the rank of $A$ is equal to 2.
6. Consider the quadratic form \( q : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
q(x, y) = 6x^2 + 10xy + 6y^2.
\]

(a) (5 points) Find a symmetric matrix \( S \) such that

\[
q(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot S \cdot \begin{bmatrix} x \\ y \end{bmatrix}.
\]

\[
S = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}
\]

The coefficients for \( x^2 \), \( y^2 \) go along the diagonal (in that order); we split up the "10" on \( xy \) evenly into the off-diagonals: 5 for \( xy \) and 5 for \( yx \).
(b) (15 points) Sketch the curve defined by \( g(x, y) = 1 \). Draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

Eigenvalues of \( S \):
\[
0 = \det \begin{bmatrix} 6 - \lambda & 5 \\ 5 & 6 - \lambda \end{bmatrix} = \lambda^2 - 12\lambda + 11 = (\lambda - 1)(\lambda - 11)
\]

Eigenvalues a.k.a. principal axes:
\[
\lambda = 1 \\
\begin{bmatrix} 5 \\ 5 \end{bmatrix}
\]
\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x + y = 0
\]
\[
\overrightarrow{V_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]
\[
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\]
\[
\lambda = 11 \\
\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}
\]
\[
\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad -x + y = 0
\]
\[
\overrightarrow{V_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

The formula of the curve in the coordinates \([\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{Z}]\) is:
\[
c_1^2 + 11c_2^2 = 1
\]

Find intercepts:
\[
c_1 = 0 \\
11c_2^2 = 1 \\
c_2 = \pm \frac{1}{\sqrt{11}}
\]
\[
\overrightarrow{c} = \pm \frac{1}{\sqrt{11}}
\]
\[
c_1 = \pm 1 \\
c_2 = 0
\]
(c) (15 points) Find the singular value decomposition for $S$. Sketch the curve obtained by applying $S$ to the set of unit vectors in $\mathbb{R}^2$ (i.e. sketch the image of the unit circle under $S$). Is this curve equal to the curve found in part (b)?

$S^*S = S^2$, so since $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors of $S$, they are eigenvectors of $S^*S$. Moreover, the corresponding eigenvalues for $S^*S$ are found by:

$S^*S \vec{v}_1 = S(S\vec{v}_1) = S(1\cdot \vec{v}_1) = S\vec{v}_1 = \vec{v}_1$

$S^*S \vec{v}_2 = S(S\vec{v}_2) = 1\lVert S\vec{v}_2 \rVert = 1\lVert \vec{v}_2 \rVert = \sqrt{2} \vec{v}_2$

thus, the singular values of $S$ are $\sqrt{1} = 1$ and $\sqrt{121} = 11$. By finding the eigenvectors of $S^*S$, we have found the matrix $U$, it is given by

$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. We compute the columns of $U$:

$S\vec{v}_1 = \sigma_1 \vec{v}_1$\hfill $S\vec{v}_2 = \sigma_2 \vec{v}_2$

$\vec{v}_1 = 1\cdot \vec{v}_1$\hfill $\lVert \vec{v}_2 \rVert = \lVert \vec{v}_2 \rVert$

$\vec{u}_1 = \vec{v}_1$\hfill $\vec{v}_2 = \vec{u}_2$

$\vec{u}_1, \vec{u}_2$ are eigenvectors of $S$.

Thus, $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} -\vec{v}_1 \\ -\vec{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} -\vec{v}_1 \\ -\vec{v}_2 \end{bmatrix}$.
The principal axes of this ellipse are the same as the principal axes of the ellipse in part (b).

This curve is definitely not equal to the curve we found in part (b). The ellipses are stretched in different directions!
(d) (3 points) Determine which symmetric matrices $S$ have the property that the image of the unit circle under $S$ and the curve $\{ v \in \mathbb{R}^2 : v \cdot S v = 1 \}$ are equal. How many such matrices are there?

There is only one: the identity matrix.

To see this, think about how to graph these curves.

* The image of the unit circle under $S$ is an ellipse (because $S$ is a linear transformation).
* $\exists v: v \cdot S v = 13$ is only an ellipse if $S$ is positive-definite, so for the curves to be the same, all eigenvalues of $S$ are positive.

We will show that they are all not only positive, but equal to 1.

Where do these curves intersect the principal axes?

* For $\exists v: v \cdot S v = 13$, these axes are spanned by eigenvectors of $S$, and the intersections are at $\left( \pm \frac{\sqrt{13}}{\lambda_i} \right)$ for the corresponding eigenvalues $\lambda_i$.
* Now, when $S$ is SYMMETRIC, the principal axes for the image of the unit circle are spanned (again) by eigenvectors of $S$ (we see this in part (c), above). The principal axes intersect this ellipse at the singular values, $\sigma_i = \lambda_i$.

So, if these curves are equal, $\lambda_i = \frac{1}{\sigma_i}$, so all the eigenvalues are 1.

But for a symmetric matrix, this means that $S$ must be the identity matrix.
7. (15 points) Determine if each of the statements below are Always True or Possibly False.

(a) True False \( \mathbb{R}^n \) is a subspace of \( \mathbb{R}^n \).
(b) True False Let \( A \) be a square matrix. The subspaces \( \ker(A) \) and \( \im(A) \) are orthogonal.
(c) True False For any non-zero vector \( v \in \mathbb{R}^n \) there is a basis \( \beta \) such that \( [v]_\beta = e_1 \).
(d) True False \( \mathbb{R}^2 \) is a subspace of \( \mathbb{R}^3 \).
(e) True False Let \( A \) be a square \( n \times n \) matrix and \( 0 \) be the \( n \times n \) zero matrix. If \( A \neq 0 \) but \( A^2 = 0 \), then \( A \) is not diagonalizable.
(f) True False Given any \( n-1 \) dimensional subspace \( V \subseteq \mathbb{R}^n \), there is an orthogonal basis \( v_1, \ldots, v_n \) of \( \mathbb{R}^n \) such that \( v_1, \ldots, v_n \notin V \).
(g) True False If \( A \) is a square matrix and \( \det(A) = 9 \), then \( A \) is not orthogonal.
(h) True False Every subspace of \( \mathbb{R}^n \) has an orthonormal basis.
(i) True False The least squares solution \( x^* \) for the matrix equation \( Ax = b \) is the solution to \( Ax = b \) for which \( \|Ax^* - b\| \) is minimized.
(j) True False Let \( A \) be a square matrix. The matrix \( I + kA \) is invertible for all but finitely many values of \( k \in \mathbb{R} \).
(k) True False Let \( A \) be a square matrix. The matrix \( A \) is diagonalizable if and only if the matrix \( A^2 \) is diagonalizable.
(l) True False If a linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) preserves area, then \( T \) is orthogonal.
(m) True False If \( A = A^T \), then \( A \) is diagonalizable.
(n) True False The span of \( k \) vectors \( v_1, \ldots, v_k \in \mathbb{R}^n \) is a \( k \)-dimensional subspace.
(o) True False If \( A \) is a \( 3 \times 3 \) matrix, then there is a line \( L \subseteq \mathbb{R}^3 \) through the origin such that if \( v \in L \) then \( Av \in L \).
(p) True False If \( a, b, c \in \mathbb{R} \) are positive numbers, then solution set to the equation \( ax^2 + bxy + cy^2 = 1 \) is an ellipse.
(q) True False The matrices \[
\begin{bmatrix}
0 & 2 \\
1 & -5 \\
0 & 4
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 \\
0 & 2 \\
0 & 0
\end{bmatrix}
\]
are similar.
(r) True False If \( A \) is a square matrix, then \( \det(A^T A) \geq 0 \).
(s) True False There exists an invertible \( 10 \times 10 \) matrix that has 92 ones among its entries.
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(a) Check directly
(b) Would be true of \( \text{im} (A^T) \) instead. Consider \( B \).
(c) Check directly
(d) The vectors have different numbers of entries
(e) All eigenvalues are 0, so if \( A = S \Omega S^{-1}, A = 0 \).
   Since \( A \neq 0 \), it is not diagonalizable. 
(f) Imagine "tilting" the orthonormal basis away from the subspace; a diagram is below.

(g) Orthogonal matrices have determinant \( \pm 1 \).
(h) This is what the Gram-Schmidt process finds.
(i) \( \lambda \) usually does not solve \( Ax = b \).
(j) Unless \( \frac{1}{k} \) is an eigenvalue of \( A \), \( I + kA \) is invertible.
(k) Consider \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \), and compare to (e).
(l) Consider \( \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \); determinant is 1; it's definitely not orthogonal.
(m) This is what the spectral theorem says.
(n) Consider \( u_1 = u_2 = \ldots = u_n = \xi_0 3 \); span is 0-dimensional.
(6) This is another way of saying that $A$ has a (real) eigenvector. But since $3$ is odd, the characteristic polynomial of $A$ is a cubic and thus has at least one real root, which is a real eigenvalue of $A$, with a real eigenvector.

(7) Consider $a = c = 1$, $b = 16$. Then the corresponding matrix is
\[
\begin{bmatrix}
1 & 8 \\
8 & 1
\end{bmatrix},
\]
and $\det \begin{bmatrix} 1-x & 8 \\ 8 & 1-x \end{bmatrix} = x^2 - 2x - 63 = (x-7)(x+9)$. Since $-7,9$ have different signs, they are the 2 eigenvalues of the matrix of the quadratic form here, the solution set is actually a hyperbola.

(8) The eigenvalues of
\[
\begin{bmatrix}
0 & 0 & 2 \\
1 & 0 & -5 \\
0 & 1 & 4
\end{bmatrix}
\]
are $1$ with algebraic multiplicity $2$, $2$ with algebraic multiplicity $1$, but the geometric multiplicity of $1$ is only $1$, so the matrix isn't diagonalizable.
(r) \[ \text{det}(A^T A) = \text{det}(A^T) \cdot \text{det}(A) = \text{det}(A)^2 \geq 0 \]

Make sense if $A$ is square

(5) No matter how the eight not-one entries are scattered, they miss at least two rows of the matrix. Thus, at least two rows of the matrix are the same, so it isn't invertible.