This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- If you use a “fundamental theorem” you must indicate this and explain why the theorem may be applied.

- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

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1. Consider the system of linear equations:

\[ \begin{align*}
2x + z &= -5, \\
3x + y + z &= -1, \\
-x &= 2.
\end{align*} \]

(a) (5 points) Express this system of linear equations as a single matrix equation \( Ax = b \).

\[
\begin{bmatrix}
2 & 0 & 1 \\
3 & 1 & 1 \\
-1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
=
\begin{bmatrix}
-5 \\
-1 \\
2
\end{bmatrix}
\]

\[
A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}
\]
(b) (15 points) Is the matrix $A$ invertible? If so, calculate $A^{-1}$.

$$
\begin{bmatrix}
2 & 0 & 1 \\
3 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix} \xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

$$
A^{-1} = \begin{bmatrix}
0 & 0 & -1 \\
-1 & 1 & -1 \\
1 & 0 & 2
\end{bmatrix}
$$

Since $\text{RREF}(A) = I$, the matrix $A$ is invertible and

$$
A^{-1} = \begin{bmatrix}
0 & 0 & -1 \\
-1 & 1 & -1 \\
1 & 0 & 2
\end{bmatrix}
$$

(c) (10 points) Find all solutions to this system of linear equations.

$$
A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b} \text{ is the unique solution.}
$$

$$
\begin{bmatrix}
0 & 0 & -1 \\
-1 & 1 & 1 \\
1 & 0 & 2
\end{bmatrix} \begin{bmatrix}
-5 \\
-1 \\
2
\end{bmatrix} = \begin{bmatrix}
-2 \\
6 \\
-1
\end{bmatrix}
$$
2. Let \( L \subseteq \mathbb{R}^2 \) be the line of slope 2 through the origin. Let \( R : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map defined by reflection over \( L \).

(a) (14 points) Determine the matrix for the transformation \( R \).

Write \( e_i = e_i^\| + e_i^\perp \) where \( e_i^\| = c_i \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] \in L \)

and \( e_i^\perp \cdot \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] = 0 \).

Then \( R e_i = e_i^\| - e_i^\perp = e_i^\| - (e_i - e_i^\|) \)

\[ = 2 e_i^\| - e_i \]

\[ c_i = \frac{e_i \cdot \left[ \begin{array}{c} 1 \\ 2 \end{array} \right]}{\left[ \begin{array}{c} 1 \\ 2 \end{array} \right] \cdot \left[ \begin{array}{c} 1 \\ 2 \end{array} \right]} = \begin{cases} 1/5 & \text{if } i = 1 \\ 2/5 & \text{if } i = 2 \end{cases} \]

So \( R e_i = \begin{cases} 2 \left( \frac{1}{5} \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] \right) - \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] & \text{if } i = 1 \\ 2 \left( \frac{2}{5} \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] \right) - \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] & \text{if } i = 2 \end{cases} \)

\[ = \begin{cases} \left[ \begin{array}{c} -3/5 \\ 4/5 \end{array} \right] & \text{if } i = 1 \\ \left[ \begin{array}{c} 4/5 \\ 3/5 \end{array} \right] & \text{if } i = 2 \end{cases} \]

The matrix for \( R \) is

\[ (R e_1, R e_2) = \left( \begin{array}{cc} -3/5 & 4/5 \\ 4/5 & 3/5 \end{array} \right) \]
(b) (7 points) Let $\theta \approx 63.43^\circ$ be the angle between $L$ and the $x$-axis. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by rotation clockwise around the origin by angle $2\theta$. Determine the matrix for the product $AR$. (Hint: think geometrically. It is not necessary to compute $A$.) Explain your answer.

To find the matrix for $AR$ we must calculate $ARE_1$ and $ARE_2$.

Observe

\[
\begin{array}{cc}
\text{reflect} & \text{rotate} \\
\text{over} & \text{clockwise} \\
L & 2\theta \\
\end{array}
\]

Notice that since rotation and reflection preserve angles $e_1 \perp e_2$, $Re_1 \perp Re_2$ and $ARE_1 \perp ARE_2$.

It follows the matrix for $AR$ is

\[
\begin{bmatrix}
ARE_1 & ARE_2 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}.
\]

Your answer to this problem should be a matrix with real numbers as entries (not trigonometric expressions).
3. Let $k \in \mathbb{R}$ and $B_2 = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & k \end{bmatrix}$.

(a) (12 points) Find bases for the kernel and image of $B$. (Your answer may depend on $k$).

Row reduce $B$:

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & k \end{bmatrix} \xrightarrow{\text{II} - 2I} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & k-4 \end{bmatrix} \Rightarrow \begin{cases} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & k-4 \end{bmatrix} & k = 4 \\ \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} & k \neq 4 \end{cases}$$

So,

$$\text{Row min}(B) = \begin{cases} \begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } k = 4 \\ \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{if } k \neq 4 \end{cases}$$

Two cases:

Case I $k = 4$:

- Image of $B$ has basis $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$.
- Kernel of $B$ has basis $x = -\frac{1}{3}y - \frac{2}{3}z$.

Case II $k \neq 4$:

- Image of $B$ has basis $\begin{bmatrix} 3 \\ 6, \begin{bmatrix} 2 \end{bmatrix} \end{bmatrix}$.
- Kernel of $B$ has basis $x = -\frac{1}{3}y$.

(b) (4 points) State the rank-nullity theorem and verify it for $B$.

The rank nullity theorem states:

If $A: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation,

$$m = \dim(\mathbb{R}^m) = \dim(\text{Im}(A)) + \dim(\text{Ker}(A))$$

$$= \text{Rank}(A) + \text{Nullity}(A).$$

For $A = B$, we have:

$$3 = \frac{2}{\text{rank}} + \frac{1}{\text{nullity}}, \quad \text{if } k \neq 4, \quad 3 = \frac{1}{\text{rank}} + \frac{2}{\text{nullity}}, \quad \text{if } k = 4.$$
(c) (10 points) Consider the set of relations between the rows of $B$:

$$V := \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2 : c_1 \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & k \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \right\}.$$ 

Show that $V$ is a subspace of $\mathbb{R}^2$ and calculate its dimension.

**Answer**

Observe that $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in V$ iff

$$3c_1 + 6c_2 = 0,$$

$$c_1 + 2c_2 = 0,$$

$$2c_1 + kc_2 = 0.$$

1. The first relation says that $V$ is a subset of the line through the origin of slope 2.
2. The second relation says the same.
3. The third relation says $V$ is a subset of the line through the origin of slope $\frac{k}{2}$.

$*$ $V$ is the intersection of these three lines. $*$

If $k \neq 4$, the lines in (1) and (3) intersect only at the origin and we see that

$$V = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is a subspace, which has dimension 0.}$$

If $k = 4$, $V$ is the line of slope 2 through the origin so $V$ is a subspace and dimension of $V$ is 1.
(c) (10 points) Consider the set of relations between the rows of B:

\[ V := \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2 : c_1 \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 6 & 2 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \right\}. \]

Think Kernel.

Show that \( V \) is a subspace of \( \mathbb{R}^2 \) and calculate its dimension.

Note that

\[ c_1 \begin{bmatrix} 3 \\ 12 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ if and only if} \]

\[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 3c_1 + 2c_2 \\ c_1 \\ c_2 \end{bmatrix} \]

So \( V \) is the Kernel of \( \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 1 & k \end{bmatrix} \),

and hence it is a subspace. (Alternatively, one could check the three properties of a subspace.)

The dimension of \( V \) is 0 if \( \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 0 \\ k \end{bmatrix} \) are linearly independent (i.e. not scalar multiples of each other).

This happens when \( k \neq 4 \).

When \( k = 4 \), we observe that \( \text{Im} \left( \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \right) \)

is the line spanned by \( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \), so

\[ \dim (\text{kernel}) = 2 - \dim (\text{Image}) = 1. \]
(c) (10 points) Consider the set of relations between the rows of $B$:

$$V := \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2 : c_1 \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 6 & 2 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \right\}.$$

Show that $V$ is a subspace of $\mathbb{R}^2$ and calculate its dimension.

\[ \text{(worth 3 points)} \]

We check that $V$ satisfies properties of a subspace:

1. Contains 0 vector:

   $[0\ 0\ 0] \in V$ because $0 \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + 0 \begin{bmatrix} 6 & 2 & k \end{bmatrix} = [0\ 0\ 0]$.

2. Closed under scalar multiplication:

   If $[c_1\ c_2] \in V$, then
   
   $$c_1 \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 6 & 2 & k \end{bmatrix} = c \left( c_1 \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 6 & 2 & k \end{bmatrix} \right)$$
   
   $$= c \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$
   
   $$= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

   So $[c_1\ c_2] \in V$.

3. Closed under addition:

   If $[c_1\ c_2], [c'_1\ c'_2] \in V$, then
   
   $$(c_1 + c'_1) \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + (c_2 + c'_2) \begin{bmatrix} 6 & 2 & k \end{bmatrix}$$
   
   $$= c_1 \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 6 & 2 & k \end{bmatrix} + c'_1 \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} + c'_2 \begin{bmatrix} 6 & 2 & k \end{bmatrix}$$
   
   $$= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$
(d) (2 points) Verify that the dimension of the space of relations between the rows of $B$ (i.e. \( \dim(V) \)) equals the number of rows of zeros in \( \text{rref}(B) \) (i.e. the number of rows which don’t contain a pivot). Can you give a rationale why this equality should hold true for a general matrix?

From (a),

\[
\text{rref}(B) = \begin{cases} 
\begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } k = 4 \\
\begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } k \neq 2.
\end{cases}
\]

So the \# of rows of zeros is

\[
\begin{cases} 
1 & \text{if } k = 4 \\
3 & \text{if } k \neq 2.
\end{cases}
\]

\[= \dim(\text{rref}(B)) = \dim(\text{space of relations between rows}).\]

In general,

a non-trivial relation between the rows of a matrix gives (a sequence) of elementary row operations resulting in a zero row. For example, the relation \([-2] \] between the rows of $B$ (when $k = 4$) allows for the production of a zero row by the row operation \( R_2 \leftarrow 2R_1 \). On the other hand, a sequence of row operations results in a zero row gives rise to a relation between the rows. Hence

\[ \dim(V) > 0 \Leftrightarrow \# \text{ of zeros} > 0 \]

or equivalently

\[ \dim(V) = 0 \Leftrightarrow \# \text{ rows is 0} \]
The equality
\[ \dim (V) = 0 \iff \# \text{zero rows is } 0 \]
can be promoted to an equivalence
\[ \dim (V) = n \iff \# \text{zero rows is } n \]
by thinking about how the right and left hand side changes as one deletes rows of the matrix $B$. 
4. (6 points) How does the rank of a square matrix \( A \) compare to the rank of \( A^2 := AA \)? For each of the three inequalities below, determine if there is a square matrix \( A \) so that the inequality is satisfied:

1. \( \text{rank}(A) < \text{rank}(A^2) \), \( \text{No} \)
2. \( \text{rank}(A) = \text{rank}(A^2) \), \( \text{Yes} \)
3. \( \text{rank}(A) > \text{rank}(A^2) \), \( \text{Yes} \)

Explain your answer.

\( \text{Claim: } \text{rank}(A) \geq \text{rank}(A^2) \) \( \text{So 1 is false} \)

why?

\[ \text{Im}(A^2) = \{ A^2x : x \in \mathbb{R}^n \} = \{ Av : v = Ax \in \text{Im}(A) \} \subseteq \{ Av : v \in \mathbb{R}^n \} = \text{Im}(A) \]

So the image of \( A^2 \) is a subspace of the image of \( A \).

It follows

\[ \text{rank}(A^2) = \dim(\text{Im}(A^2)) \leq \dim(\text{Im}(A)) = \text{rank}(A) \]

\( \text{Claim: } \text{If } A = I, \text{ then } A^2 = I \). So \( \text{rank}(A) = \text{rank}(A^2) \).

\( \text{Claim: } \text{There is no } A \text{ such that } \text{rank}(A) > \text{rank}(A^2) \).

what is it? To have such an \( A \), we need "information loss" between applying \( A \) once and applying \( A \) twice. In linear algebra terms, we need to find a matrix \( A \) which contains a vector which is both in its kernel and image. \( \alpha \text{(image contained in } x\text{-axis)} \)

Take \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). Then \( A \) has \( \text{rank } 1 \)

(\( \text{Kernel contains } x\text{-axis} \))

\( A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) has \( \text{rank } 0 \).