(Practice) Midterm 1

(1) Find all solutions in \( \mathbb{R}^4 \) to the system of linear equations:

\[
\begin{align*}
3x_1 + 2x_2 + x_3 + x_4 &= 1, \\
2x_2 + x_4 &= -1, \\
x_1 - 2x_4 &= 0.
\end{align*}
\]

(2) Let \( V \subseteq \mathbb{R}^3 \) be the plane (through 0) consisting of solutions to the equation \( x + y - z = 0 \).
   (a) Find the orthogonal projections of the standard basis vectors
      \[
      e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
      \]
      onto \( V \).
   (b) Let \( \text{Proj}_V : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation defined by orthogonal projection onto \( V \). Find a \( 3 \times 3 \) matrix \( A \) such that \( Ax = \text{Proj}_V(x) \) for all \( x \in \mathbb{R}^3 \).
   (c) Calculate \( A^2 \).
   (d) Observe that \( A^2 = A \). Can you explain this phenomenon geometrically?

(3) Is the matrix \( A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \) invertible?

(4) Let \( B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \).
   (a) Find a basis for the image of \( B \).
   (b) Find a basis for the kernel of \( B \).
   (c) State the rank-nullity theorem. Verify that this theorem holds for \( B \).
   (d) Are there two linearly independent vectors that lie outside the image of \( B \)? If so, give examples of such a pair of vectors. If not, explain why.