(Practice) Midterm 1

(1) Find all solutions in $\mathbf{R}^4$ to the system of linear equations:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + x_4 &= 1, \\ 2x_2 + x_4 &= -1, \\ x_1 - 2x_4 &= 0. \end{aligned}$$

(2) Let $V \subseteq \mathbf{R}^3$ be the plane (through 0) consisting of solutions to the equation $x + y - z = 0$.
   (a) Find the orthogonal projections of the standard basis vectors

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   onto V.
   (b) Let $\mathrm{Proj}_V : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by orthogonal projection onto V. Find a 3×3 matrix A such that $Ax = \mathrm{Proj}_V(x)$ for all $x \in \mathbf{R}^3$.
   (c) Calculate A^2.
   (d) Observe that $A^2 = A$. Can you explain this phenomenon geometrically?

(3) Is the matrix $A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ invertible?

(4) Let $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

   (a) Find a basis for the image of B.
   (b) Find a basis for the kernel of B.
   (c) State the rank-nullity theorem. Verify that this theorem holds for B.
   (d) Are there two linearly independent vectors that lie outside the image of B? If so, give examples of such a pair of vectors. If not, explain why.