Lecture 9: Solving $A\vec{x} = \vec{b}$.

Problem: Let $A = \begin{bmatrix} -6 & 2 & 14 \\ -9 & 3 & 21 \end{bmatrix}$. For which $\vec{b} \in \mathbb{R}^2$ does $A\vec{x} = \vec{b}$ have a solution? For each such $\vec{b}$ find all solutions $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$.

Solution: Let's brainstorm some $\vec{b}$'s for part I.

$\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a solution.

$\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2$ is a solution.

$\vec{b} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$ then $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution.

More generally Ae_i is the ith column of A.

So, each of the columns of this matrix is a $\vec{b}$.

Producing b's is easy just plug in vectors into Ax.

In general,

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\left(x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 6 \\ 1 \end{bmatrix} \right)$$

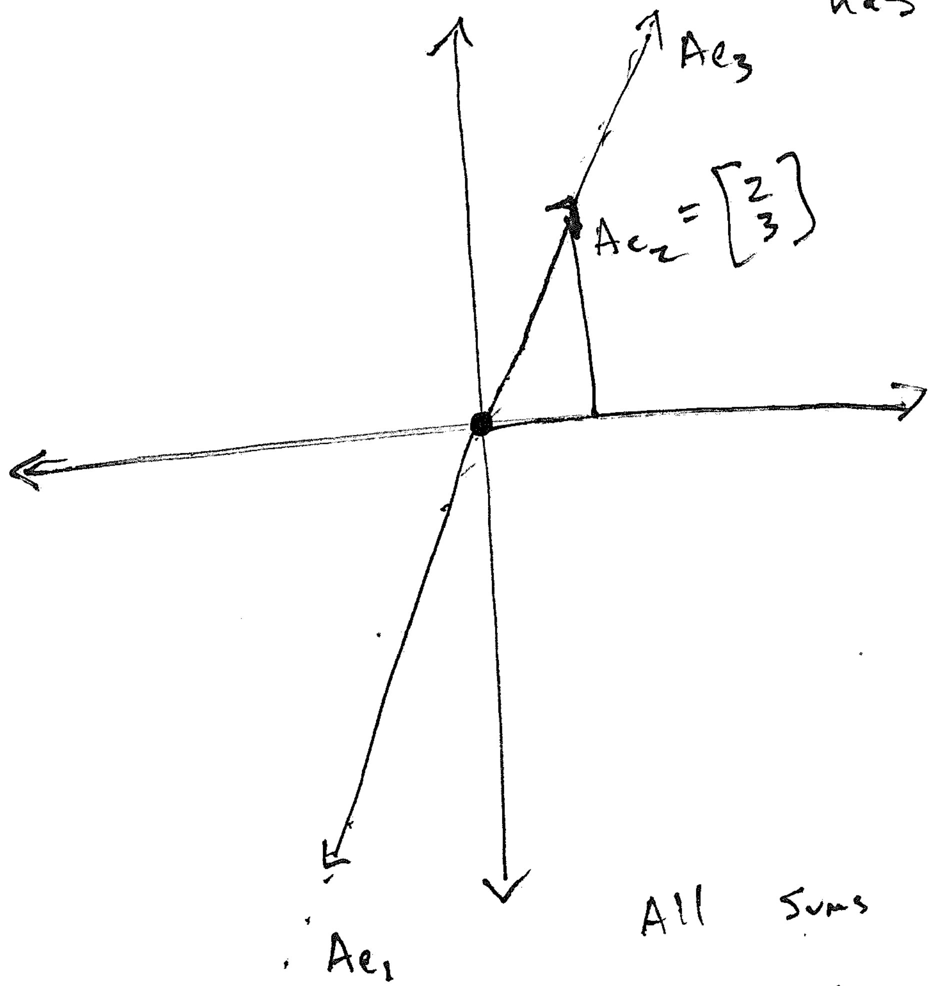
by linearity

$$= x_1 A e_1 + x_2 A e_2 + x_3 A e_3.$$

$$= x_1 \begin{bmatrix} -6 \\ -9 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 14 \\ 21 \end{bmatrix}.$$

The b's for which $Ax = b$ has a solution are obtained by multiply columns of A by constants and adding.

These are the b's for which $Ax = b$ has a solution



All sums of the form

$x_1 Ae_1 + x_2 Ae_2 + x_3 Ae_3$

lie on the line of slope $3/2$ through 0.

$\vec{b}$ for which solutions exist are exactly those on the line of slope $3/2$ through 0

Warm Up. (Originally presented at the beginning)

★ Let $A = \begin{bmatrix} -6 & 2 & 14 \\ -9 & 3 & 21 \end{bmatrix}$.

Find all solutions $\vec{x} \in \mathbb{R}^3$ to

$$A\vec{x} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution: To solve form augmented matrix $\left[A \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix}\right]$ row reduce. As I row reduce nothing happens to $\left|\begin{matrix} 0 \\ 0 \end{matrix}\right]$. So I might as well not include this column.

$$\begin{bmatrix} -6 & 2 & 14 \\ -9 & 3 & 21 \end{bmatrix} \xrightarrow{I/-6} \begin{bmatrix} \boxed{1} & -1/3 & 7/-3 \\ -9 & 3 & 21 \end{bmatrix}$$

$$\xrightarrow{II+9I} \begin{bmatrix} 1 & -1/3 & 7/-3 \\ 0 & 0 & 0 \end{bmatrix}.$$

So solutions satisfy

$$x - \frac{1}{3}y - \frac{7}{3}z = 0.$$

Solve for x.

$$x = \frac{1}{3}y + \frac{7}{3}z.$$

Solutions are

$$\begin{bmatrix} \frac{1}{3}y + \frac{7}{3}z \\ y \\ z \end{bmatrix}$$

$$= y\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \frac{7}{3} \\ 0 \\ 1 \end{bmatrix}$$

for any choice of $y, z \in \mathbb{R}$.

Solution set is a plane through 0.

Part II: For each $\vec{b}$ in this line we want to solve $A\vec{x} = \vec{b}$.

Warm up: We saw solutions to $Ax = \vec{0}$ are

$$K = \left\{ y\begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \frac{7}{3} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Plane through 0.

Exercise: Find all $\vec{x}$ such that $Ax = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Form augmented matrix

Row reduce

$$\left[\begin{array}{ccc|c} -6 & 2 & 14 & 2 \\ -9 & 3 & 21 & 3 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & 7/-3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

get 0 in second step.

Extracting we have

$$x - \frac{1}{3}y - \frac{7}{3}z = -\frac{1}{3}.$$

Solve for x and we have solutions are

$$\begin{bmatrix} \frac{1}{3}y + \frac{7}{3}z - \frac{1}{3} \\ y \\ z \end{bmatrix}$$

Solutions are

$$\left\{ \underbrace{y\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \frac{7}{3} \\ 0 \\ 1 \end{bmatrix}}_{\substack{\text{Solutions to} \\ Ax=0}} + \underbrace{\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix}}_{\substack{\text{A solution} \\ Ax=b.}} \right\}.$$

Solutions to $Ax = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a plane.
Moreover its the translate of K by $\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$.

More generally, if $\vec{b}$ lies on the line of slope $\frac{3}{2}$ then

$$\vec{b} = c\begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ for some } c.$$

So if $\vec{x}$ is a solution to

$$A\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then by linearity

$$A(c\vec{x}) = cA\vec{x} = c\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solutions

$$\left\{ \underbrace{\overset{y'}{\cancel{cy}}\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + \overset{z'}{\cancel{cz}}\begin{bmatrix} \frac{7}{3} \\ 0 \\ 1 \end{bmatrix}}_{K} + \underbrace{\begin{bmatrix} -\frac{c}{3} \\ 0 \\ 0 \end{bmatrix}}_{\text{A particular solution}} \right\}$$

Translate to $Ax = 0$ by a solution.

Geometrically, solutions to $Ax = b$ for every b that has a solution is a plane and that plane is a fixed translate of the plane K solving $Ax = 0$.

This case is representative of solving $Ax = b$ in general. What we'll do now is give things names.

Def: Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.

1. The kernel of A is the set of vectors $\vec{x} \in \mathbb{R}^m$ such that

$$A\vec{x} = \vec{0}.$$

(i.e. solutions of this equation)

It's denoted $\mathrm{Ker}(A)$.

(2) The image of A is the set of vectors $\vec{b} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$ has a solution (they're the vectors that are mapped to under A).

Denoted $\mathrm{Im}(A)$.

Def: Let $\vec{x}_1, \ldots, \vec{x}_k \in \mathbb{R}^n$ be a list of vectors. The span of $\vec{x}_1, \ldots, \vec{x}_k$ are all vectors of the form

$$c_1 \vec{x}_1 + c_2 \vec{x}_2 + \ldots + c_k \vec{x}_k$$

where $c_1, \ldots, c_k$ are constants.

Thm: Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and $e_1, \ldots, e_m$ be the vectors

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th place}$$

then $\mathrm{Im}(A) = \mathrm{Span}(Ae_1, Ae_2, \ldots, Ae_m)$.

If some one asks you what the image of A is its okay (but not great) to say

$$\mathrm{Span}(Ae_1, \ldots, Ae_m).$$

Thm: If $A: \mathbb{R}^m \to \mathbb{R}^n$ is a liner transformation then the solution set of $Ax = b$ for $b \in \mathrm{Im}(A)$ has the form

$$\vec{x}_0 + \ker(A).$$

$$= \{\vec{x}_0 + \vec{k} \mid k \in \ker(A)\}$$

where $\vec{x}_0$ is <u>any</u> vector which solves $A\vec{x}_0 = \vec{b}$.

Next time we'll discuss what a kernel and Image can look like.