Lecture 8: Invertible Matrices

Problem: Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ be an arbitrary vector. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find all solutions $\vec{x} \in \mathbb{R}^2$ to the equation $A\vec{x} = \vec{b}$ (as functions of b_1 and b_2).

Solution: Recall $A\vec{x} = \vec{b}$ is the matrix equation which encodes the system of linear equations

$$\begin{aligned} 2x + y &= b_1 \\ x + y &= b_2 \end{aligned}$$

We find solutions to $A\vec{x} = \vec{b}$ by solving this system.

① Consider augmented matrix

$$\left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 1 & 1 & b_2 \end{array}\right]$$

$$\underbrace{\hphantom{2 \quad 1}}_{A}$$

Row Reduce

$$\left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 1 & 1 & b_2 \end{array}\right] \xRightarrow{\mathrm{II} \leftrightarrow \mathrm{I}} \left[\begin{array}{cc|c} \boxed{1} & 1 & b_2 \\ 2 & 1 & b_1 \end{array}\right]$$

$$\xRightarrow{\mathrm{II} - 2\mathrm{I}} \left[\begin{array}{cc|c} \boxed{1} & 1 & b_2 \\ 0 & -1 & b_1 - 2b_2 \end{array}\right]$$

$$\xRightarrow{-\mathrm{II}} \left[\begin{array}{cc|c} \boxed{1} & 1 & b_2 \\ 0 & \boxed{1} & -b_1 + 2b_2 \end{array}\right]$$

$$\xRightarrow{\mathrm{I} - \mathrm{II}} \left[\begin{array}{cc|c} \boxed{1} & 0 & b_1 - b_2 \\ 0 & \boxed{1} & -b_1 + 2b_2 \end{array}\right].$$

Solution

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ -b_1 + 2b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}_{B = A^{-1}} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Sanity check:

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 7 \\ 2 \end{bmatrix}}_{\vec{b}} = \begin{bmatrix} 7 - 2 \\ -7 + 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Remark on the previous problem:

We may regard

$$A : \mathbb{R}^2 \to \mathbb{R}^2$$

as a function which transforms vectors according to the rule $\vec{x} \longmapsto A\vec{x}$. ~~For~~ From from this perspective the matrix B is the rule which transforms vectors back. That is, if

$$A\vec{x} = \vec{b}$$

then

$$B\vec{b} = \vec{x}.$$

Composing these maps, we have

$$(BA)\vec{x} = B(A\vec{x}) = B\vec{b} = \vec{x}$$

for all $\vec{x} \in \mathbb{R}^2$. In other words, BA is the identity matrix.

Check

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2-1 & 1-1 \\ -2+2 & -1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \checkmark$$

☆ Based on this we define: ☆

$\boxed{\begin{array}{l}\text{Def: Let } A \text{ be a square matrix. The matrix } A \text{ is called } \underline{\text{invertible}} \text{ if there} \\ \text{is a matrix } B \text{ such that} \\ \qquad BA = I \\ \text{In this case, the matrix } B \text{ is called the } \underline{\text{inverse}} \text{ of } A \text{ and is denoted } A^{-1}.\end{array}}$

☆ ☆

Remark: If A is invertible, then $A^{-1}\vec{b}$ is a (the only) solution to $A\vec{x} = \vec{b}$.

Warning! Not all (square) matrices are invertible. For example 1×1 matrices are just numbers and

$$[b][a] = [1]$$

if and only if $a \neq 0$.

$$\boxed{\text{How do you determine if } A \text{ is invertible } \underline{\text{and}} \text{ how do you compute } A^{-1}?}$$

In the case, we saw we solved

$$A\vec{x} = \vec{b}$$

in indeterminants $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \vec{b}$. We found $A^{-1} = B$ was the coeff matrix of the resulting linear equations.

This is true in general. We'll keep track of the coeff of the b's using an augmented matrix.

How do you determine if A^{-1} exists <u>and</u> how do you find it.

Procedure:

(idea is to solve $A\vec{x} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ and find formulae for solutions in terms of $b_1, \ldots, b_n$). We keep track of coeff. of these b's using an augmented matrix.

Step 1: Form the augmented matrix

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

$$b_1 \ldots b_n$$

Step 2: Row reduce. If A is invertible then Rref(A) of A is the identity <u>and</u> the resulting matrix is

$$\left[\begin{array}{c|c} \underset{\text{rref}(A)}{I} & A^{-1} \end{array}\right]$$

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Determine if A is invertible. If so compute A^{-1}.

Step 1: Form augmented matrix.

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \overset{\mathrm{I} \leftrightarrow \mathrm{II}}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right].$$

Step 2: row reduce.

$$\overset{\mathrm{II} - 2\mathrm{I}}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\overset{-\mathrm{II}}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\overset{\mathrm{I} - \mathrm{II}}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right].$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{So } A^{-1} \text{ exists}} \qquad \underbrace{\begin{matrix} 1 & -1 \\ -1 & 2 \end{matrix}}_{\text{This is } A^{-1}.}$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.}$$

Example 2: Determine if $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$

is invertible and if so find A^{-1}.

Step 1: Form augmented matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\quad A \qquad\qquad I$$

Step 2 row reduce.

(next page)

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{III}-3\text{I}]{\text{II}-2\text{I}}$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{III}-5\text{II}]{\text{I}-\text{II}} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 3 & -1 & 0 \\ 0 & \boxed{1} & 0 & -2 & 1 & 0 \\ 0 & 0 & \boxed{-1} & 7 & -5 & 1 \end{array}\right]$$

$$\xrightarrow{-\text{III}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

$$\xrightarrow{\text{I}-\text{III}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{A \text{ is invertible}} & \underbrace{\begin{matrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{matrix}}_{\text{This is } A^{-1}.} \end{array}\right]$$

Solution

$$\boxed{A^{-1} = \begin{bmatrix} 11 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}.}$$

Sanity check:

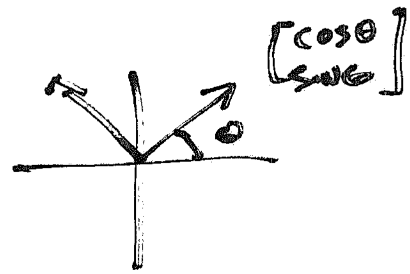
Compute that $A^{-1} \cdot A = I$.

Computing inverses of some examples from geometry.

Example 1: Rotation.

Let θ be an angle. Rotation around O by θ is given by

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



What is A^{-1}?

The inverse is rotation by $-\theta$

$$A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin(\theta) \\ -\sin\theta & \cos(\theta) \end{bmatrix}.$$

<u>Check this:</u> $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

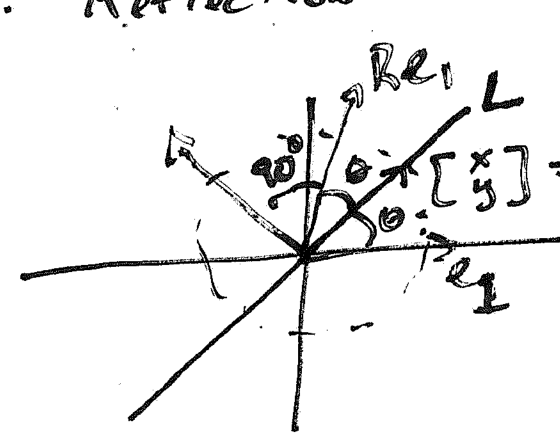
$$A \cdot A^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & b^2 + a^2 \end{bmatrix}$$

$a^2 + b^2$
$= \cos^2\theta$
$+ \sin^2\theta$
$= 1$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \blacksquare$$

Example 2: Reflection over a line in $\mathbb{R}^2$



$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$

Reflection over L is given by

$$A = \frac{1}{x^2+y^2}\begin{bmatrix} x^2-y^2 & 2xy \\ 2xy & -x^2+y^2 \end{bmatrix}.$$

What is A^{-1}?

$$A^{-1} = A.$$

Un reflect by reflecting back!

Let's check this in the special case that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

In this case

$$A = \frac{1}{1}\begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -\cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$ then

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$ and

$$A \cdot A^{-1} \stackrel{?}{=} A \cdot A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & b^2 + a^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ since $a^2 + b^2 = \underline{1}$.