Lecture 6: Dot Products, orthogonal projections, ~~and Matrix Multiplication.~~

Def: Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

be a pair of vectors in $\mathbb{R}^n$. The dot product of $\vec{x}$ and $\vec{y}$ is the quantity

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Example:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix} = 1 \cdot 7 + 2 \cdot 2 + 3 \cdot 1$$
$$= 14.$$

Why do we care about dot products?

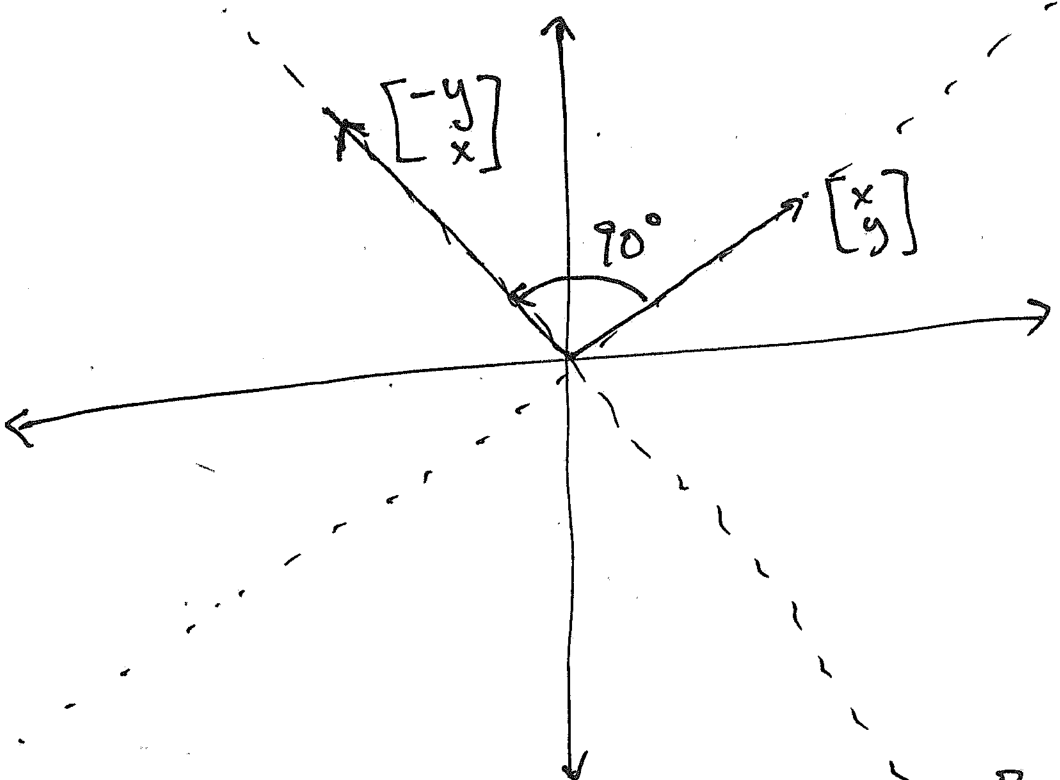
★ Dot products can be used to detect if vectors are perpendicular to each other.

Orthogonal = perpendicular to each other.

$$\boxed{\text{Thm: Let } \vec{x} \text{ and } \vec{y} \text{ be vectors in } \mathbb{R}^n \text{ then } \vec{x} \text{ and } \vec{y} \text{ orthogonal if and only if } \vec{x} \cdot \vec{y} = 0}$$

Example:

Scalar multiples are parallel



Perpendicular to $\begin{bmatrix}x\\y\end{bmatrix} \Longleftrightarrow$ multiples of $\begin{bmatrix}-y\\x\end{bmatrix}$.

$$\left(c\begin{bmatrix}-y\\x\end{bmatrix}\right)\cdot\begin{bmatrix}x\\y\end{bmatrix} = c(-yx) + c(xy)$$
$$= 0.$$

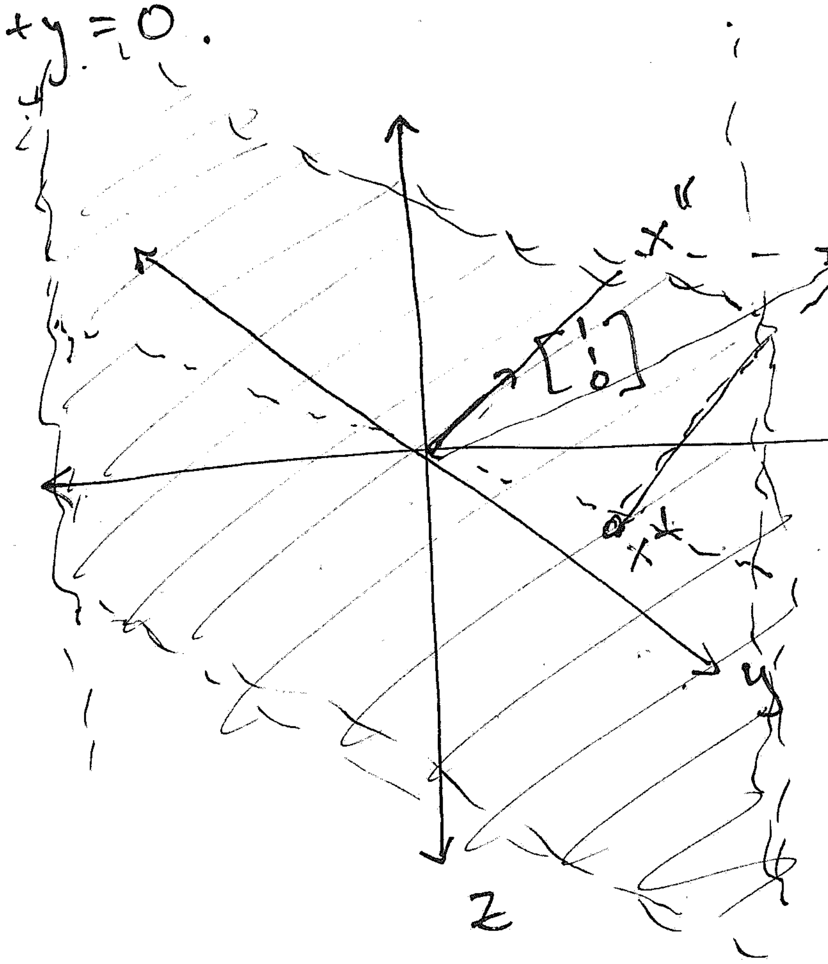
Example $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. IF $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

is perpendicular to v then

$$0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{x + y}.$$

In other words $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution

to $x + y = 0$.



Solution Set: Vertical plane above the Line of Slope -1 in x, y plane through 0

In general, the solutions to

$$ax + by + cz = 0$$

(where a, b, c are fixed numbers)

is the plane perpendicular to $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Algebraic Properties of Dot Products

① $x \cdot y = y \cdot x$

Dot product is liner in y

② $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$

③ $x \cdot (c\,y) = c\,(x \cdot y) \qquad c \in \mathbb{R}$.

Last time we considered the problem of finding the parallel and perpendicular vectors $x^{\parallel}$ and $x^{\perp}$ to some line such that a given vector $x = x^{\parallel} + x^{\perp}$.

Using dot products we can give another solution to this problem and generalize it to higher dimensions.

★ Problem: Let $v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

Find vectors $x^{\parallel}$ and $x^{\perp}$ such that $x^{\parallel}$ is parallel to v and $x^{\perp}$ is perpendicular to v and $x = x^{\parallel} + x^{\perp}$

($x^{\parallel}$ and $x^{\perp}$ depend on v)

Solution:

Parallel means that

$$x^{\parallel} = cv \quad \text{for some scalar } c$$

$\cancel{x^{\perp}}$

Perpendicular means

$$x^{\perp} \cdot v = 0.$$

Want $x = x^{\parallel} + x^{\perp}$.

Consider

$$x \cdot v = (x^{\perp} + x^{\parallel}) \cdot v$$

$$= \cancel{x^{\perp} \cdot v}^{\,0} + x^{\parallel} \cdot v$$

$$= (cv) \cdot v$$

$$= c\,(v \cdot v)$$

Solving for c we have

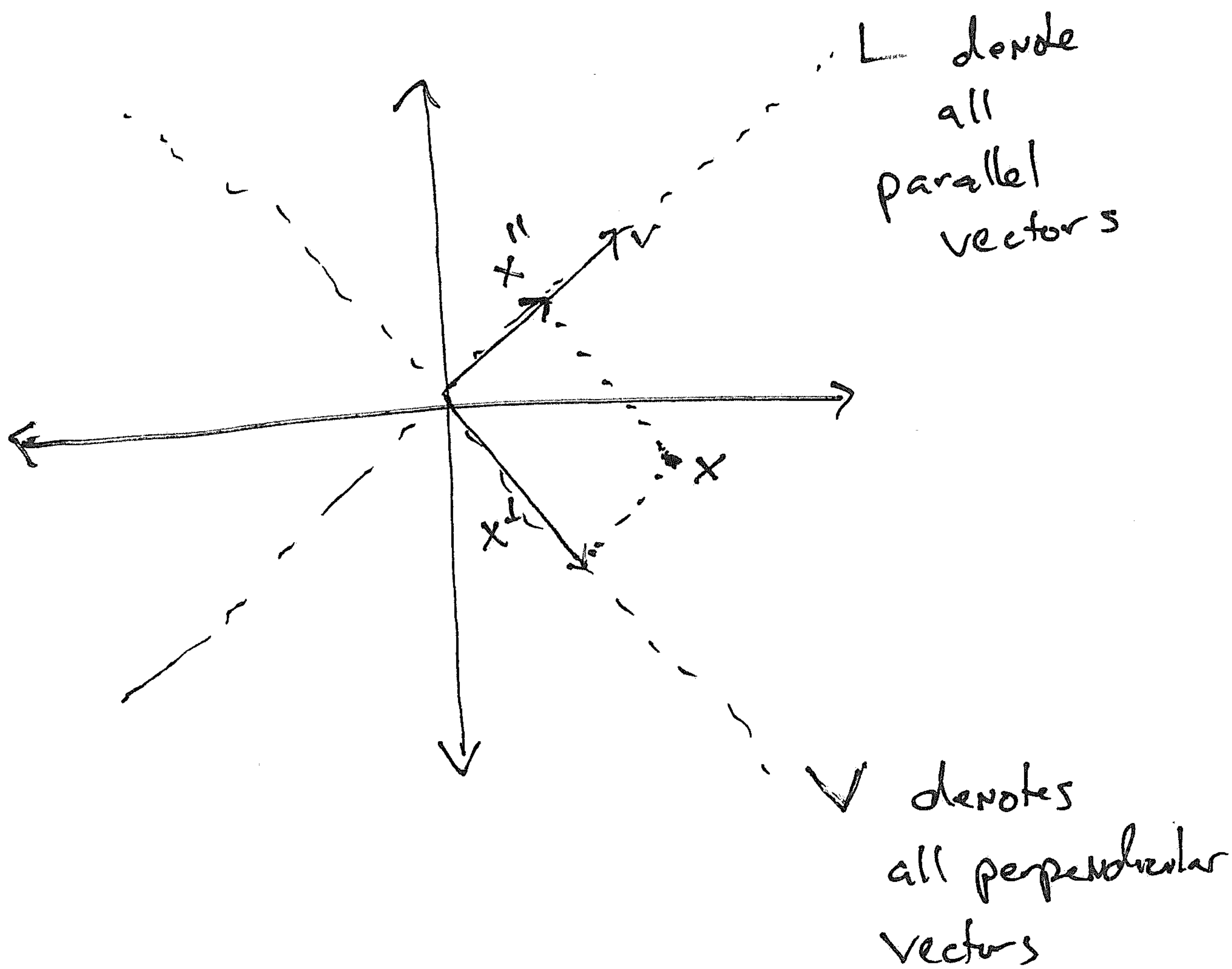
$$c = \frac{x \cdot v}{v \cdot v} = \frac{7 \cdot 1 + 1 \cdot 3 + 0 \cdot 1}{1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0}$$

$$= \frac{10}{2} = 5.$$

$$\boxed{x^{\parallel} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = 5v.}$$

$$\boxed{x^{\perp} = x - x^{\parallel} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.}$$

Check $x^{\perp} \cdot v = 0$



Def: Given a vector v, the maps

$$\mathrm{Proj}_L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

given

$$x \longmapsto x^{\parallel}$$

and

$$\mathrm{Proj}_V : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$x \longmapsto x^{\perp}$$

are linear transformations $\left(\text{i.e. } (x+y)^{\parallel} = x^{\parallel} + y \atop \vdots \right)$

(depend on $\vec{v}$).

The maps Proj_L and Proj_V are called orthogonal projections onto L and V respectively.

Some properties: ① if $x \in V$ then

$$\text{Proj}_V(x) = x.$$

(Same with $\text{Proj}_L(x) = x$ if $x \in L$).

② $\text{Proj}_V(x) \in V$ for all $x \in \mathbb{R}^n$

(Same with Proj_L)

Problem: Find the matrix for the orthogonal projection onto the plane $x+y=0$ in $\mathbb{R}^3$.

Solution: This problem looks a little confusing. What happened to the parallel + perpendicular vectors?

Let's unpack this.

$$\text{plane } x+y=0 \longleftrightarrow \begin{array}{c}\text{Set of vectors}\\ \text{perpendicular}\\ \text{to}\\ \begin{bmatrix}1\\1\\0\end{bmatrix}.\end{array}$$

Projection onto this plane is given by

$$x \longmapsto x^{\perp}$$

where $x^{\perp}$ is the part of x perpendicular to the vector $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

Solution is a 3 by 3 matrix

To find the matrix we consider the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Multiply a 3×3 by matrix by e_i picks out i^{th} column of the matrix.

Desired matrix

$$\begin{bmatrix} | & | & | \\ \mathrm{Proj}_{\vec{v}}\, e_1 & \mathrm{Proj}_{\vec{v}}\, e_2 & \mathrm{Proj}_{\vec{v}}\, e_3 \\ | & | & | \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Calculate:

$$e_1^{\parallel} = \frac{e_1 \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$e_2^{\parallel} = \frac{e_2 \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$e_3^{\parallel} = \frac{e_3 \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{0}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Calculate:

$$\mathrm{Proj}(e_1) = e_1^{\perp} = e_1 - e_1^{\parallel} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\mathrm{Proj}(e_2) = e_2^{\perp} = e_2 - e_2^{\parallel} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\mathrm{Proj}(e_3) = e_3^{\perp} = e_3 - \vec{0} = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Depends on v

This matrix tells you how to take a vector x and produce $x^\perp$.

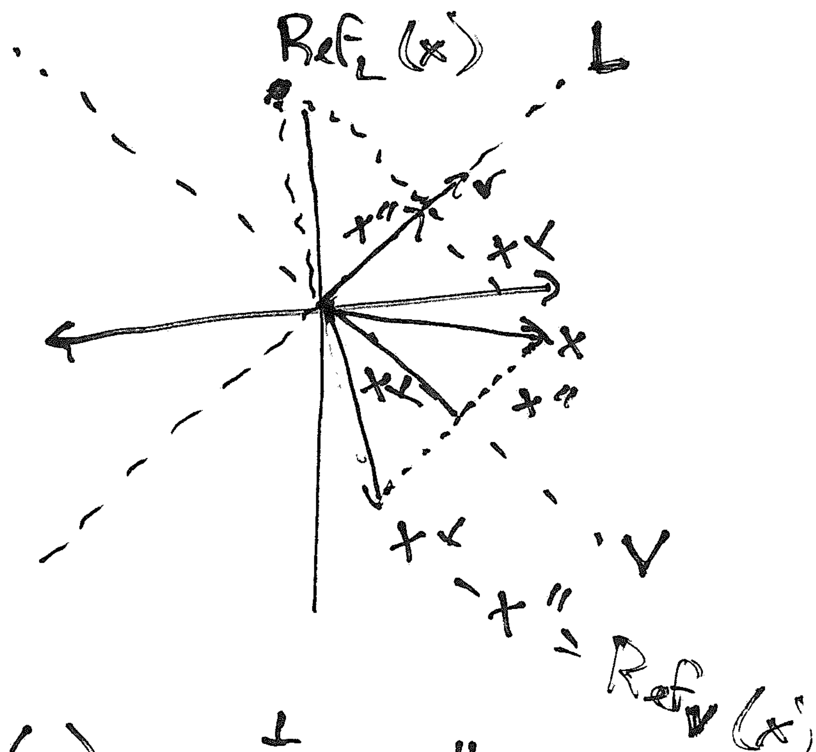
Example: $x = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

$$A(x) = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = x^\perp$$

This is what we found before.

There are two other linear transformations you can construct using parallel and perpendicular vectors. These are the reflections.



$$Ref_V(x) = x^{\perp} - x^{\parallel}$$

$$Ref_L(x) = -x^{\perp} + x^{\parallel}.$$