Lecture 5: Examples of Linear Transformations from geometry.

Last time we defined a linear transformation as a map

$$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

such that

① $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^m$

② $T(cx) = cT(x)$ for all scalars $c \in \mathbb{R}$ and vectors $x \in \mathbb{R}^m$.

★ Multiplication by a matrix is a Linear transformation; all L.T. arise in this way.

Geometric Interpretation of 1 and 2

(I) T maps parallelograms (with a corner at the origin) to parallelograms (with a corner at the origin.

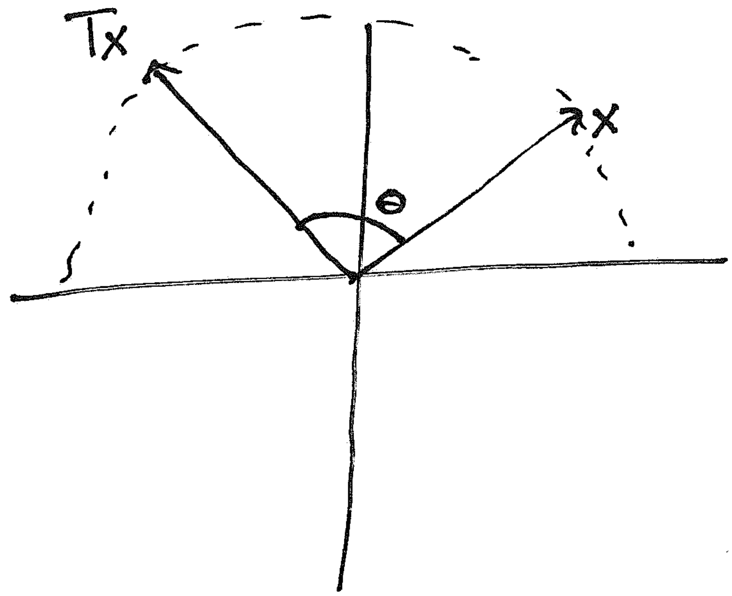
(II) T maps lines through the origin to lines through the origin.

Goal for today: Find / Describe some maps satisfing (I) and (II) <u>and</u> calculate the corresponding matrix.

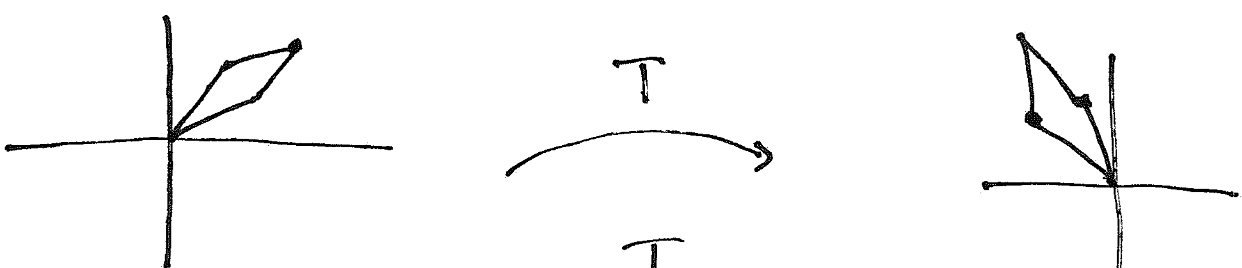
Example 1: Rotation

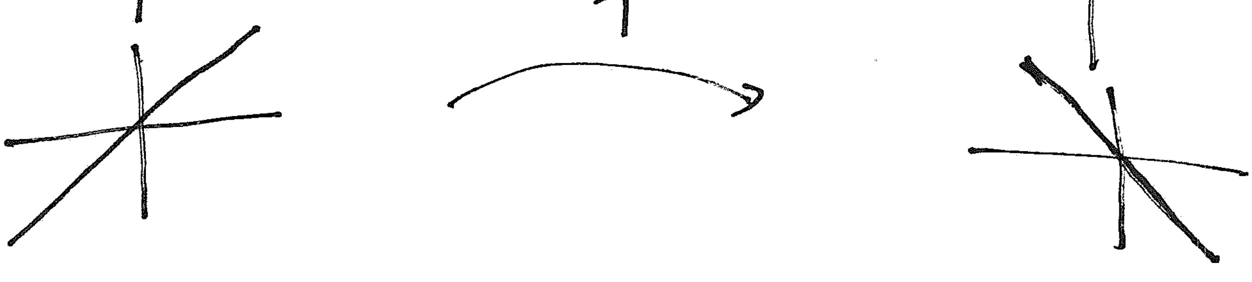
Let θ be an angle (in radians) in the range $0 \leq \theta \leq 2\pi$

Consider the map $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation (around 0) by angle θ.



This map satisfies (I) and (II)





So T is a linear transformation.

What is the corresponding matrix?

- It's a square 2 by 2 matrix

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- Observe that if

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then $$Ae_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

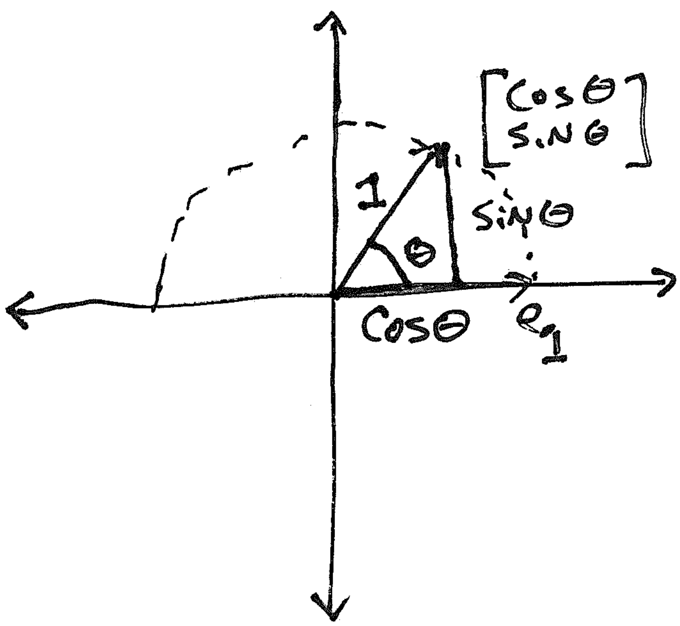
$$Ae_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

So T corresponds to

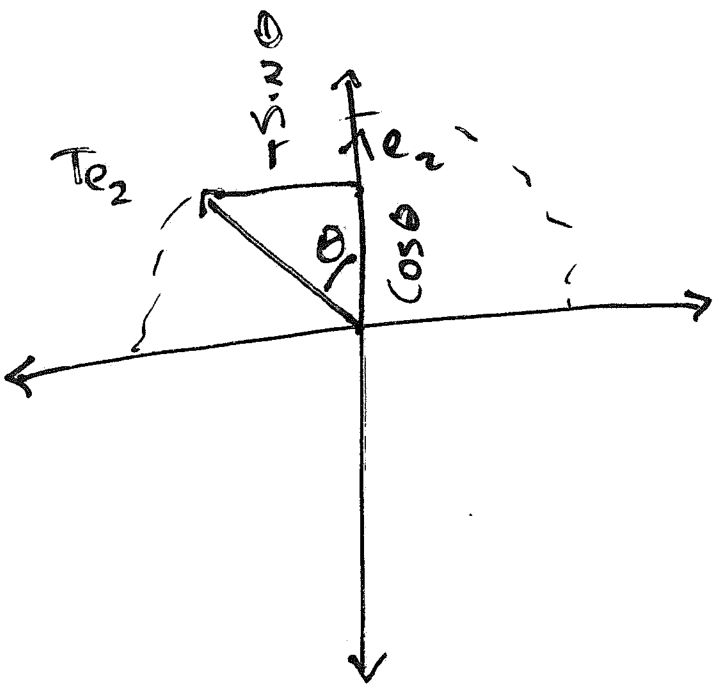
$$\begin{bmatrix} | & | \\ Te_1 & Te_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ Ae_1 & Ae_2 \\ | & | \end{bmatrix}.$$

To calculate the corresponding matrix we need to calculate

$$Te_1 \quad \text{and} \quad Te_2 .$$



$$Te_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} .$$



$$Te_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Rotation by θ is given by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

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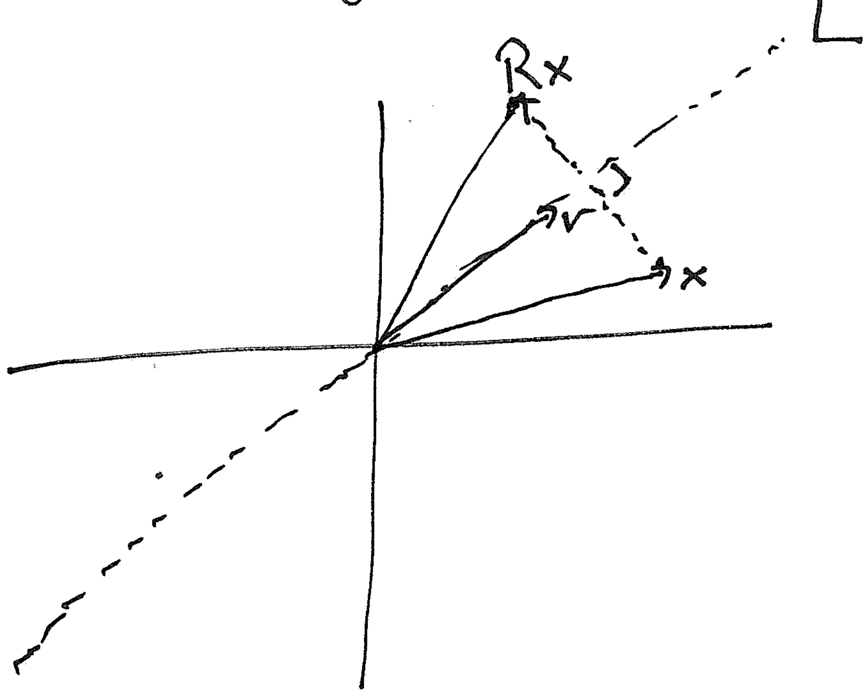
1. We checked T was linear
   (we did this by showing parallelograms and lines went to the same)

2. To write down the matrix calculate Te_1 and Te_2.

Example 2: Reflection in the plane through a line (going through the origin).

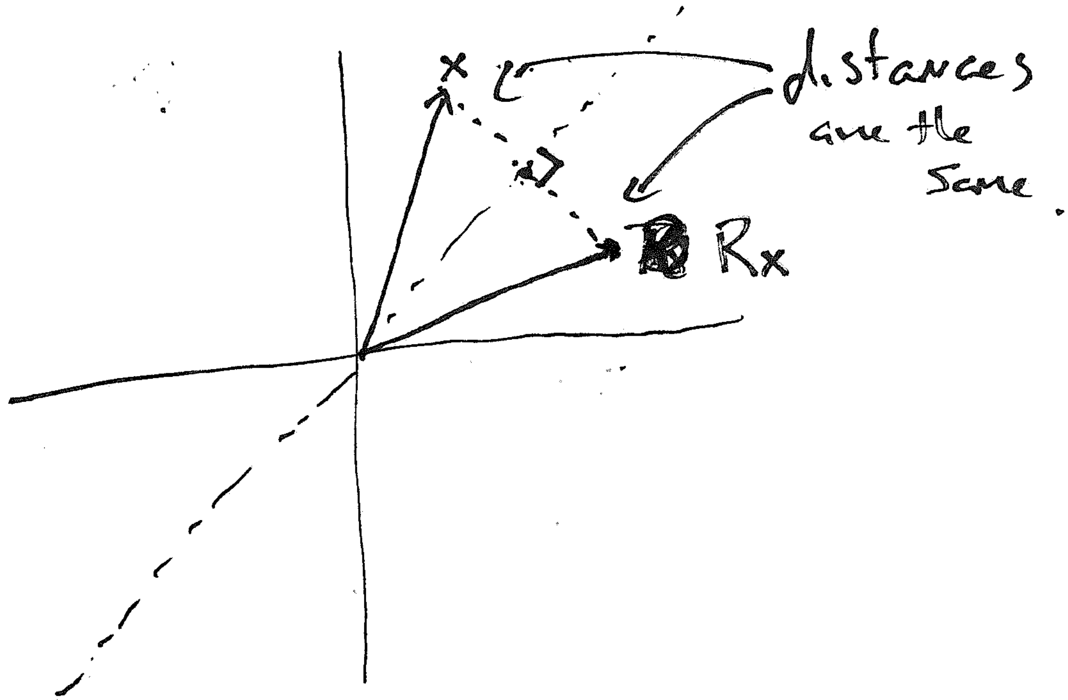
Let $v \in \mathbb{R}^2$ be a vector. Consider the line containing v.



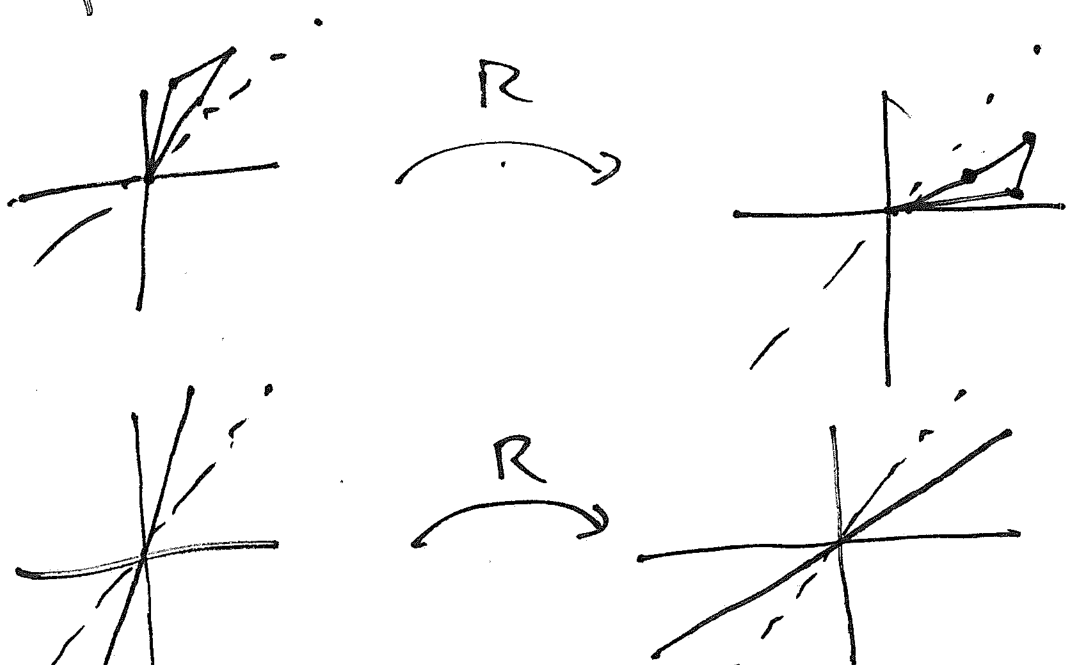
Reflection through L is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$

$$R : \mathbb{R}^2 \to \mathbb{R}^2.$$

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by reflection through this line.

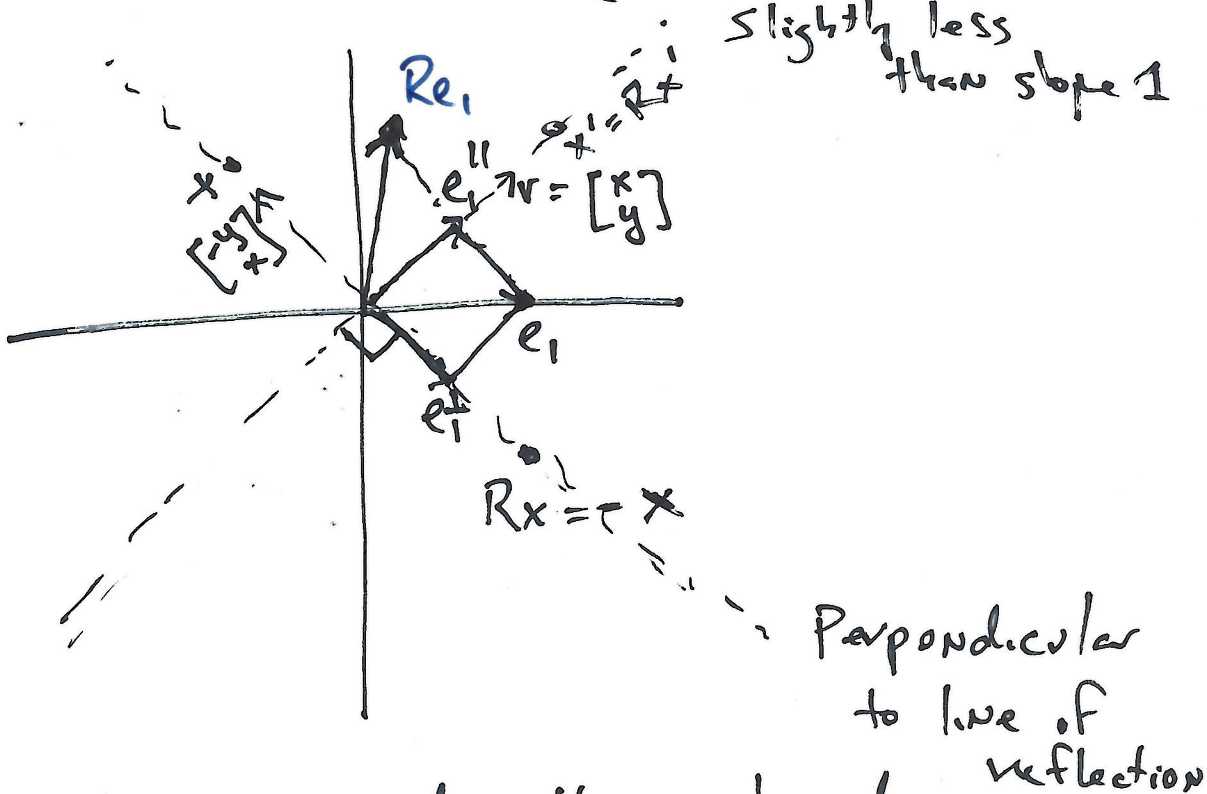


The map R satisfies (I) and (II)



So R is linear!

What is the corresponding matrix?



★ Calculating Re_1 directly is hard.

But for those vectors x contained in the line of reflection, we have $Rx = x$ (parallel to v)

vectors perpendicular to the line of reflection are also easy to calculate the reflection for.

$$Rx = -x.$$

To calculate Re_1, we will express e_1 as the sum of a parallel vector and perpendicular vector.

Then Given a decomposition

$$e_1 = e_1^{\perp} + e_1^{\parallel}.$$

$$\begin{aligned} Re_1 &= R(e_1^{\perp} + e_1^{\parallel}) \\ &= Re_1^{\perp} + Re_1^{\parallel} \\ &= -e_1^{\perp} + e_1^{\parallel}. \end{aligned}$$

To calculate Re_1, we need to find $e_1^{\parallel}$ and $e_1^{\perp}$.

Assume that $v = \begin{bmatrix} x \\ y \end{bmatrix}$ for $x, y \in \mathbb{R}$.

Parallel vectors look like multiples of v

$$e_1^{\parallel} = a \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{for some } a \in \mathbb{R}.$$

Perpendicular vectors are ~~the~~ obtained by rotation by 90° from a parallel vector.

Perpendicular

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation } 90^\circ}.$$

All perpendicular vectors for v are of the form

$$e_1^{\perp} = b\begin{bmatrix} -y \\ x \end{bmatrix} \quad \text{for some } b \in \mathbb{R}.$$

We have to find a and b such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} x \\ y \end{bmatrix} + b\begin{bmatrix} -y \\ x \end{bmatrix}$$

$$e_1 = e_1^{\parallel} + e_1^{\perp}$$

Example. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find a, b

$$\begin{bmatrix} 2a-b \\ a+2b \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

System of linear equations in unknows a and b.

Solution:

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1/5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -1/5 \end{array}\right]$$

$a = 2/5$, $b = -1/5$.

In general, row reducing the matrix

$$\left[\begin{array}{cc|c} x & -y & 1 \\ y & x & 0 \end{array}\right]$$

we find

$$a = \frac{x}{x^2+y^2} \quad \text{and} \quad b = \frac{-y}{x^2+y^2}.$$

So:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{\frac{x}{x^2+y^2}\begin{bmatrix} x \\ y \end{bmatrix}}_{e_1^{\parallel}} + \underbrace{\frac{-y}{x^2+y^2}\begin{bmatrix} -y \\ x \end{bmatrix}}_{e_1^{\perp}}$$

$$R\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{x}{x^2+y^2}\begin{bmatrix} x \\ y \end{bmatrix} + \frac{y}{x^2+y^2}\begin{bmatrix} -y \\ x \end{bmatrix}.$$

$$= \begin{bmatrix} \dfrac{x^2-y^2}{x^2+y^2} \\[2ex] \dfrac{2xy}{x^2+y^2} \end{bmatrix}.$$

~~AX Sim.~~

Calculation shows:

$$R\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\dfrac{2xy}{x^2+y^2}\\[2ex] \dfrac{-x^2+y^2}{x^2+y^2}\end{bmatrix}$$

Answer:

Corresponding Matrix is

$$\begin{bmatrix}\dfrac{x^2-y^2}{x^2+y^2} & \dfrac{2xy}{x^2+y^2}\\[2ex] \dfrac{2xy}{x^2+y^2} & \dfrac{-x^2+y^2}{x^2+y^2}\end{bmatrix}.$$