Lecture 5: Examples of Linear Transformations
From geometry.

Last time we defined a linear transformation as a map

\[ T: \mathbb{R}^m \rightarrow \mathbb{R}^n \]

such that

1) \[ T(x + y) = T(x) + T(y) \]
   for all \( x, y \in \mathbb{R}^m \)

2) \[ T(cx) = c T(x) \] for all scalars \( c \in \mathbb{R} \) and vectors \( x \in \mathbb{R}^m \).

* Multiplication by a matrix is a linear transformation; all L.T. arise in this way.
Geometric Interpretation of 1 and 2

(I) $T$ maps parallelograms (with a corner at the origin) to parallelograms (with a corner at the origin).

(II) $T$ maps lines through the origin to lines through the origin.

Goal for today: Find / Describe some maps satisfying (I) and (II) and calculate the corresponding matrix.
Example 1: Rotation

Let $\theta$ be an angle (in radians) in the range $0 \leq \theta \leq 2\pi$.

Consider the map $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation (around 0) by angle $\theta$.

This map satisfies (I) and (II).
So $T$ is a linear transformation.

What is the corresponding matrix?

- It's a square 2 by 2 matrix

\[
A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}
\]

- Observe that if

\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

then \( A e_1 = \begin{bmatrix} a \\ b \end{bmatrix} \)

\( A e_2 = \begin{bmatrix} c \\ d \end{bmatrix} \)

So $T$ corresponds to

\[
\begin{bmatrix} Te_1 & Te_2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} Ae_1 & Ae_2 \\ 1 & 1 \end{bmatrix}
\]
To calculate the corresponding matrix we need to calculate $T_{e_1}$ and $T_{e_2}$.

\[
T_{e_1} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}
\]

\[
T_{e_2} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}
\]
Rotation by $\theta$ is given by the matrix

$$
\begin{bmatrix}
cos \theta & -sin \theta \\
sin \theta & \cos \theta
\end{bmatrix}
$$

or equivalently

$$
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
$$

where

$$
\begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
cos \theta \\
sin \theta
\end{bmatrix}
$$

1. We checked $T$ was linear
   (we did this by showing parallelograms and lines went to the same)

2. To write down the matrix $T_1$ and $T_2$, calculate
Example 2: Reflection in the plane through a line (going through the origin).

Let $v \in \mathbb{R}^2$ be a vector. Consider the line containing $v$.

Reflection through $L$ is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$:
$R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. 
Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by reflection through this line.

The map $R$ satisfies (I) and (II).

So $R$ is linear.
To calculate $Re$, we will express it as the sum of a parallel vector and a perpendicular vector.

$x = -x.$

But for these vectors perpendicular to the line of reflection, we have $Rx = x$.

Calculating $Re$, directly is hard. Parallel to line of reflection.

What is the corresponding matrix?
Then given a decomposition
\[ e_1 = e_1^+ + e_1^- . \]

\[ R e_1 = R (e_1^+ + e_1^-) \]
\[ = R e_1^+ + R e_1^- \]
\[ = - e_1^+ + e_1^- . \]

To calculate \( R e_1 \), we need to find \( e_1^- \) and \( e_1^+ \).

Assume that \( \mathbf{v} = [x] \) for \( x \in \mathbb{R} \).
Parallel vectors look like multiples of \( \mathbf{v} \)
\[ e_1^- = a [x] \] for some \( a \in \mathbb{R} \).

Perpendicular vectors are obtained by rotating \( e_1 \) by 90° from a parallel vector.
Perpendicular

\[
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.
\]

rotation $90^\circ$.

All perpendicular vectors are of the form

\[ e^* = b \begin{bmatrix} -y \\ x \end{bmatrix} \]

for some $b \in \mathbb{R}$.

We have to find $a$ and $b$ such that

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} x \\ y \end{bmatrix} + b \begin{bmatrix} -y \\ x \end{bmatrix}
\]

\[ e_i = e_i^* + e_i^\perp \]
Example: \[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

Find \( a, b \)

\[
\begin{bmatrix}
2a - b \\
a + 2b
\end{bmatrix}
= a \begin{bmatrix}
2 \\
1
\end{bmatrix} + b \begin{bmatrix}
-1 \\
2
\end{bmatrix}
= \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

System of linear equations in unknowns \( a \) and \( b \).

Solution:

\[
\begin{bmatrix}
2 - 1 & 1 \\
1 & 2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 2 & 6 \\
2 - a & 1
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 2 & 0 \\
0 & 2 - a & 1
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & -1/5
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 0 & 2/5 \\
0 & 1 & -1/5
\end{bmatrix}
\]

\( a = \frac{2}{5}, \ b = -\frac{1}{5} \).
In general, row reducing the matrix

\[
\begin{bmatrix}
x \\
y \\
x - y \\
x^2 + y^2 \\
o
\end{bmatrix}
\]

we find

\[
a = \frac{x}{x^2 + y^2} \quad \text{and} \quad b = \frac{-y}{x^2 + y^2}.
\]

So:

\[
\left[ \begin{array}{c}
e^{11} \\
e^{10}
\end{array} \right] = \frac{x}{x^2 + y^2} \left[ \begin{array}{c}
x \\
y
\end{array} \right] + \frac{-y}{x^2 + y^2} \left[ \begin{array}{c}
x \\
y
\end{array} \right]
\]

\[
R\left[ \begin{array}{c}
1 \\
1
\end{array} \right] = \frac{x}{x^2 + y^2} \left[ \begin{array}{c}
x \\
y
\end{array} \right] + \frac{y}{x^2 + y^2} \left[ \begin{array}{c}
-x \\
y
\end{array} \right].
\]

\[
= \begin{bmatrix}
\frac{x^2 - y^2}{x^2 + y^2} \\
\frac{2xy}{x^2 + y^2}
\end{bmatrix}
\]
Calculation shows:

\[ R^\circ [1] = \begin{bmatrix} z \times y \\
\frac{x^2 \times y^2}{x^2 + y^2} \\
-x^2 \times y^2 \\
\frac{x^2 \times y^2}{x^2 + y^2} \end{bmatrix} \]

Answer:

Corresponding matrix is

\[ \begin{bmatrix} \frac{x^2 - y^2}{x^2 + y^2} & 2 \times y & 2 \times y \\
\frac{x^2 - y^2}{x^2 + y^2} & \frac{x^2 \times y^2}{x^2 + y^2} & \frac{x^2 \times y^2}{x^2 + y^2} \\
2 \times y & \frac{x^2 \times y^2}{x^2 + y^2} & -x^2 + y^2 \\
2 \times y & \frac{x^2 \times y^2}{x^2 + y^2} & -x^2 + y^2 \end{bmatrix} \]