Lecture 5: Examples of linear transformations from geometry.
Last time, we defined linear transformations.

**Def:** A **linear transformation** is a map

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

such that

1. \( T(x+y) = T(x) + T(y) \)
   for all \( x, y \in \mathbb{R}^n \)

2. \( T(cx) = c \cdot T(x) \)
   for all \( x \in \mathbb{R}^n \) and \( c \in \mathbb{R} \).
Why do we care about linear transformations?

Linear transformations are exactly the functions defined by multiplying by a matrix i.e.

\[ T(x) = \begin{bmatrix} a_{11} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & a_{nn} \end{bmatrix} \times \]

for some matrix. This is a matrix.
What do 1 and 2 mean geometrically?

1. \( T \) maps parallelograms to parallelograms: \( T(x + y) = Tx + Ty \).
2. $T$ maps lines through the origin to lines through the origin. All multiples of $x$ of $T(x)$. All multiples of $T(x')$. Domain
Range
Goal: Find / Describe some maps satisifying 1 and 2 and calculate the corresponding matrix.
Examples of functions
Satisfy $\mathcal{O}_1 + \mathcal{O}_2$

Rotation around origin

$\mathcal{O}_1$

$\mathcal{O}_2$
Reflection through origin

Scale all vectors by $c$
Reflect our line through origin.

This is a linear transformation.
Translation is not a linear transformation.

But not through origin. (not satisfied)
Given a linear transformation, how does one find a matrix $A$ such that

$$T\mathbf{x} = A\mathbf{x}?$$
Example 1: Rotation

Let \( \theta \) be an angle in the range \( 0 \leq \theta \leq 2\pi \).

Consider the map

\[ T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

given by rotation around \( 0 \) by angle \( \theta \).
This map satisfies $\circ 1 + \circ 2$

So $T$ is a linear transformation.
What matrix does $T$ correspond to?

- $T$ corresponds to a $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

- Observe that if $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Ae_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = e_1.$$ 

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Ae_2 = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = e_2$. 


\[ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]

\[ A e_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}. \]

Second column of \( A \).

To calculate \( A \)

we must determine

\[ A e_1 = T e_1, \]

and

\[ A e_2 = T e_2. \]
Determine $T_{e_1}$:

$$\mathbf{e}_1 = [b]$$

Unit circle

$$q = \mathbf{e}_1 \cos \theta$$

$$c \theta = \sin \theta$$

$$T_{e_1} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$
Determine $T_2$:

$T_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Rotate back $90^\circ$

d = cos $\theta$

b = -sin $\theta$
\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

(counterclockwise)

Rotation by \( \theta \) is given by

by

\[ x \rightarrow Ax. \]

\[ T_x \]

\[ O \]
Summary:

Given a map \( T: \mathbb{R}^2 \to \mathbb{R}^2 \)

(I) Check that \( T \) is a linear transformation.

(II) We can write down the matrix of \( T \) by calculating \( T(e_1), T(e_2) \) (columns of the corresponding matrix).
Fix \( n \).

Def: For each \( i = n \)

let \( e_i = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 1 \\ 0 \end{bmatrix} \)

(i_{th} coordinate)

(has all entries 0 except 1 in i_{th} coordinate).

The vectors

\( e_1, \ldots, e_n \)

are called the standard basis vectors for \( \mathbb{R}^n \).
Thm: let $T : \mathbb{R}^n \to \mathbb{R}^n$
be a linear transformation and

$$A = \begin{bmatrix}
T e_1 & T e_2 & \cdots & T e_n
\end{bmatrix}$$

where $e_1, \ldots, e_n$ are the standard basis vectors for $\mathbb{R}^n$. Then

$$T x = A x \quad \text{for all } x \in \mathbb{R}^n.$$