Lecture 4: Vectors, Matrices, and Linear Transformations.

Def: A <u>vector</u> is an array of n numbers

$$\vec{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The set of all vectors of height n is denoted by $\mathbb{R}^n$.

$\in$

Notation: We will write "$\vec{x} \in \mathbb{R}^n$" to mean $\vec{x}$ is an <u>element of</u> $\mathbb{R}^n$.

Example: $\vec{0} := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ zero vector.

More generally, the zero vector is any vector all whose entries are 0.

★ Solutions to systems of linear equations are vectors.

What can you do with vectors?

① (Scalar Multiplication) Given a $c \in \mathbb{R}$, you can multiply all entries of a vector x by c.

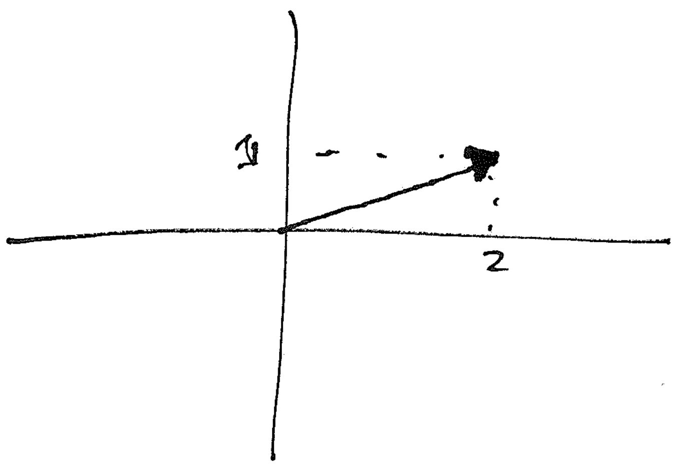
$$cx = c\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =: \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}.$$

② (Add vectors)

$$\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\text{both in } \mathbb{R}^n.} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

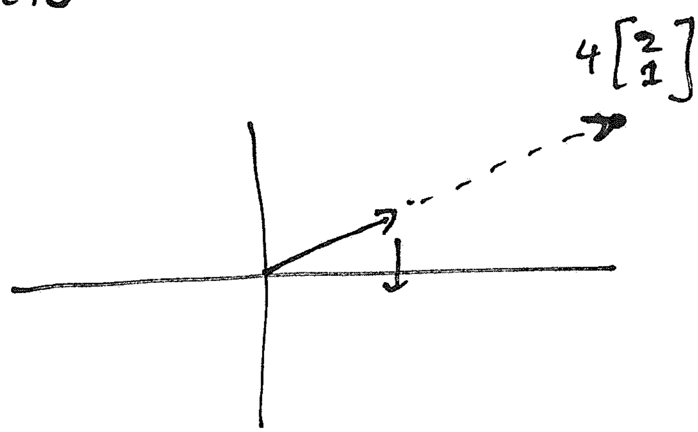
A vector may be represented geometrically as an arrow extending from the origin.

Example



$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

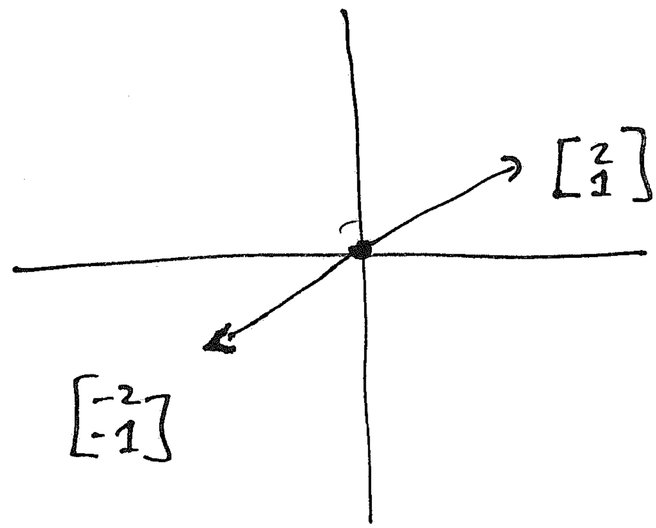
Scaler multiplication by $c > 0$, scales this vector



$$4x = \begin{bmatrix} 4 \cdot 2 \\ 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

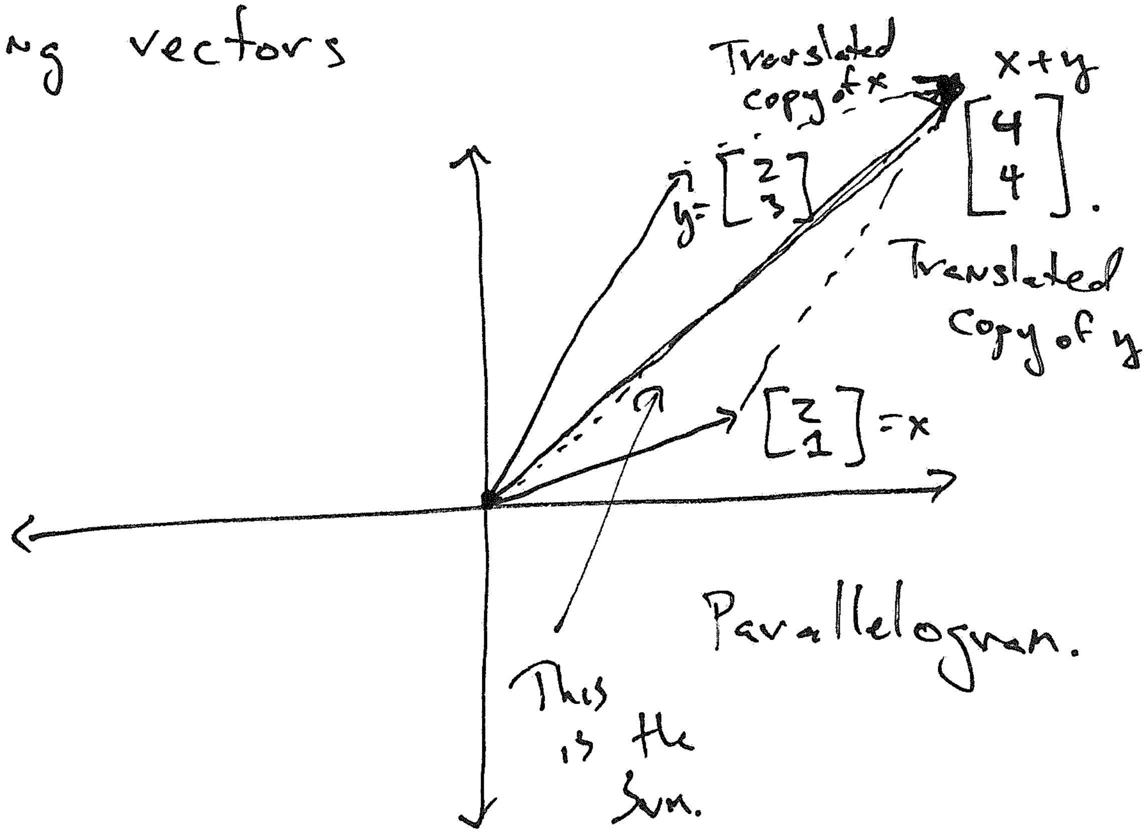
Scaler multiplication stays on same line.

Scaler multiplication by $c = -1$



Reflection through origin.

Adding vectors



A new way to think about linear systems.

Consider:

$$\begin{aligned} 7x_1 + 3x_2 + 4x_3 &= 25 \\ 2x_1 + 0x_2 + x_3 &= 5. \end{aligned} \qquad (1)$$

The left hand side is a <u>rule</u> for taking $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ and producing a vector

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \overset{\text{we went}}{=} \begin{bmatrix} 25 \\ 5 \end{bmatrix} \in \mathbb{R}^2.$$

We can think of the left hand side as a <u>function</u> $T: \mathbb{R}^3 \to \mathbb{R}^2$.

Example (Evaluating T at a vector x)

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad T = \begin{bmatrix} 7 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix}$$

Flip, Multiply coordinatewise, add.

$$T\vec{x} = \underbrace{\begin{bmatrix} 7 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix}}_{T}\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 7\cdot 1 + 3\cdot 2 + 4\cdot 3 \\ 2\cdot 1 + 0\cdot 2 + 1\cdot 3 \end{bmatrix}$$

Plug into system to evaluate

$$= \begin{bmatrix} 25 \\ 5 \end{bmatrix}.$$

Since, This is the vector we wanted $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a solution.

Example (of matrix multiplication)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 2 \\ 3\cdot 1 + 4\cdot 2 \\ 5\cdot 1 + 6\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}.$$

(Defines a function $T: \mathbb{R}^2 \to \mathbb{R}^3$)

Every matrix defines a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ (for some values of m and n) but not all functions arise in this way.

Non-example: All matrix mult sends 0 vector to 0 vector, so

$$(x, y) \longmapsto (x^2+1, y)$$

is not given by a matrix.

What are the properties of functions given by multiplication by a matrix?

The functions that arise from matrices are all <u>linear transformations</u>.

Def: A function

$$T: \mathbb{R}^m \to \mathbb{R}^n$$

is called a linear transformation if

① $T(x+y) = T(x) + T(y)$

for all $x, y \in \mathbb{R}^m$.

② $T(cx) = cT(x)$

for all $x \in \mathbb{R}^m$ and $c \in \mathbb{R}$.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 \\ 3 \cdot 1 \\ 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

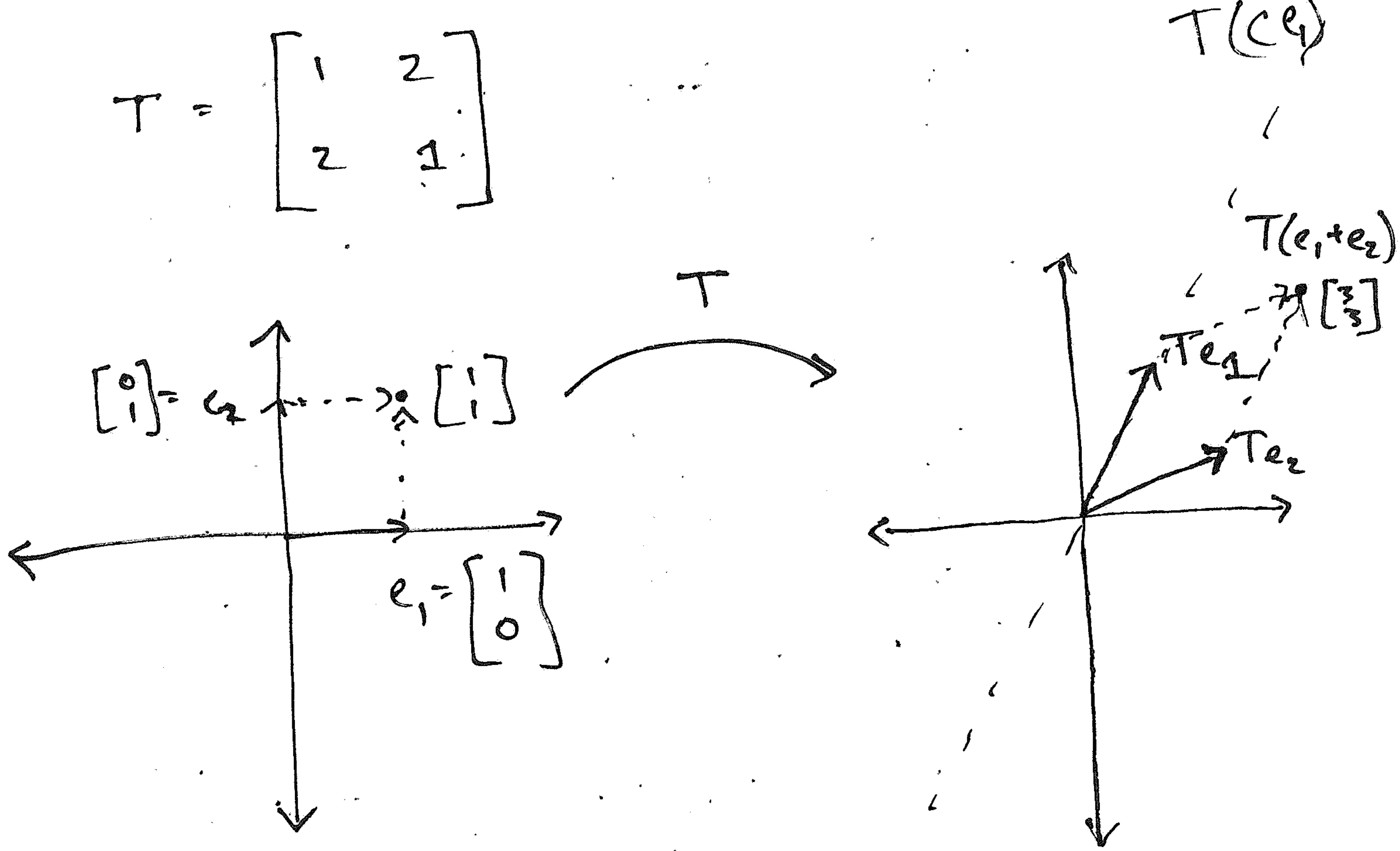
$$+$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 \\ 4 \cdot 2 \\ 6 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

Conversely, all linear transformations are given by multiply by a matrix.

What happens to vectors when multiplied by a matrix (geometrically)?

$$T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$



$$Te_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 0\cdot 2 \\ 2\cdot 1 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

picks out first column.

$$Te_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Linearity means

Sums go to sums $\Longleftrightarrow$ Parallelograms go to Parallelograms

Scalar multiples go to scalar multiples $\Longleftrightarrow$ Line through the origin goes to lines through the origin.

Example

x-axis goes to line of slope 2.

Thm: Linear transformations are <u>exactly</u> the functions which arise as multiplication by a matrix.

What is the matrix?

Consider the vector

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \begin{array}{l} 0 \text{ everywhere and} \\ 1 \text{ in } i\text{th entry} \end{array}$$

If A is a matrix

$A e_i$ is the ith column of i

In general if T is any linear transformation

$$T \longrightarrow \begin{bmatrix} Te_1 & Te_2 & \cdots & Te_n \end{bmatrix}$$

is a matrix such that the associated function equals T.