Lecture 34

SVD revisited

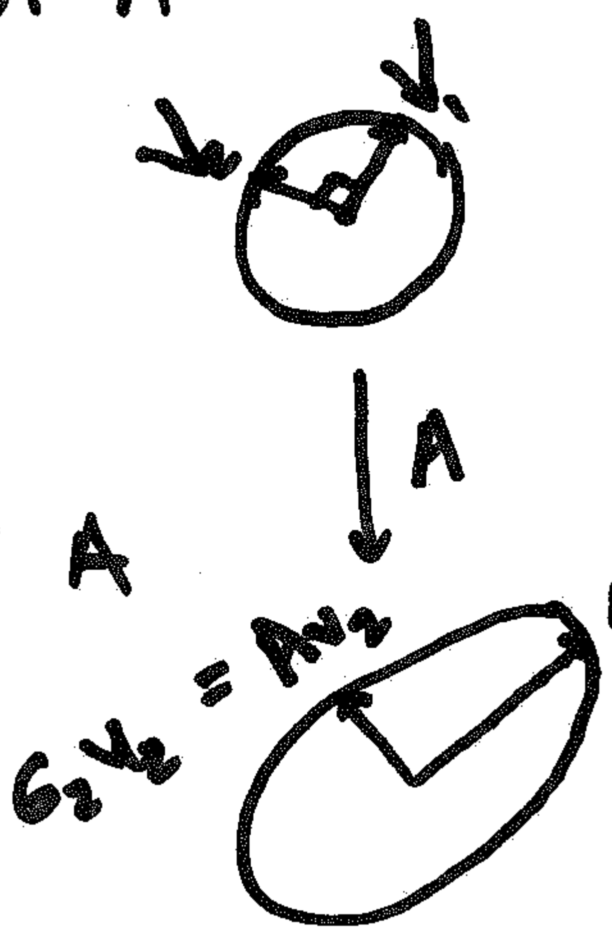
SVD

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} - v_1 - \\ - v_2 - \end{bmatrix}.$$

★ The SVD encodes which ellipse the unit circle maps to.

Domain of A



$\{\cos\theta v_1 + \sin\theta v_2 : \theta \in \mathbb{R}\}$

Range of A

$Av_1 = \sigma_1 u_1$

1. σ_1, σ_2 are the lengths of the vectors on the ellipse which lie on the principal axes

2. u_1, u_2 directions of principal axes

3. v_1, v_2 orthonormal basis of $\mathbb{R}^2$ such that Av_1, Av_2 lie on ~~pr~~principal axes

~~Image ellipse This~~

(4) Unit circle is given by

$$\{\cos\theta\, v_1 + \sin\theta\, v_2 : \theta \in \mathbb{R}\}$$

The image ellipse is

$$\{\cos\theta\,(Av_1) + \sin\theta\,(Av_2) : \theta \in \mathbb{R}\}$$

equiv

$$\{\cos\theta\,(\sigma_1 u_1) + \sin\theta\,(\sigma_2 u_2) : \theta \in \mathbb{R}\}.$$

Procedure for finding SVD

① To find V, Σ consider A^TA and its orthogonal diagonalization.

If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ are the eigenvalues of A^TA then $\sigma_i = \sqrt{\lambda_i}$ $\quad i \leq \min(n,m)$

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \sigma_3 & & \\ & 0 & & \ddots & \\ & & & & \sigma_{\min(n,m)} \end{pmatrix}$$

$n \times m$ matrix

V is the orthogonal matrix such that

$$A^T A = V D V^T,$$

where

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} = \Sigma^T \Sigma .$$

(2) Find $U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$

<u>First</u>

$$u_i = \frac{Av_i}{\sigma_i} \quad \text{if } \sigma_i \neq 0,$$

and $i \leq \min(n, m)$.

<u>Then</u>

For remaining u_i: ~~let it be the largest rule.~~ Let

$u_{k+1}, \dots, u_n$

be the ones you want.

Choose these vectors to be an orthonormal basis for $\mathrm{Im}(A)^{\perp}$.

<u>Then</u> for remaining indices

$k+1, \ldots, n$

Choose $u_{k+1}, \ldots, u_n$

to be <u>any</u> orthonormal

basis for $\mathrm{Im}(A)^\perp$.

Note: $\mathrm{Im}(A)^\perp = \mathrm{Span}(Av_1, \ldots, Av_m)^\perp$

$= \mathrm{Span}(\sigma_1 u_1, \ldots, \sigma_k u_k)^\perp$

$= \mathrm{Span}(u_1, \ldots, u_k)^\perp$.

<u>Example</u>: $A = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$.

Compute the SVD of A.

Step 1: Consider A^TA and orthogonally diagonalize.

$$A^TA = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}.$$

eigenvalues: $0, 8$.

Singular values:

! Order matters.

$$\sigma_1 = 2\sqrt{2}, \quad \sigma_2 = 0.$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

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~~ZZE~~ To find V we need an ~~orthog~~ orthonormal eigenbasis for A^TA.

$v_1 = e_2$ is an eigenvector w/ eigenvalue 8.

$v_2 = e_1$ is an eigenvector w/ eigenvalue 0.

$$V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T$$

$$= \quad V \quad D \quad V^T.$$

Step 2: FIND U.

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{2\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

u_2 a basis for

$$\mathrm{Im}(A)^\perp = \mathrm{Span}(u_1)^\perp$$

$$u_2 \in \mathrm{Span}(u_1)^\perp$$

$$= \mathrm{Ker}\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\right)$$

$$= \mathrm{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

<u>Choose</u>:

$$u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$\operatorname{Ker}\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\right) = \operatorname{Span}(u_1)^{\perp}$$

$$\shortparallel$$

$$\operatorname{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$A = U \Sigma V^{T}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{T}.$$