Lecture 33

Singular Value Decomposition

Thm: Let A be an $n \times m$ matrix, then there exists a factorization

$$A = U \Sigma V^T$$

where

- U is an $n \times n$ orthogonal matrix
- V is an $m \times m$ orthogonal matrix
- Σ is "almost diagonal"

→ ij entry of Σ is 0 if $i \neq j$

Σ is an $n \times m$ matrix.

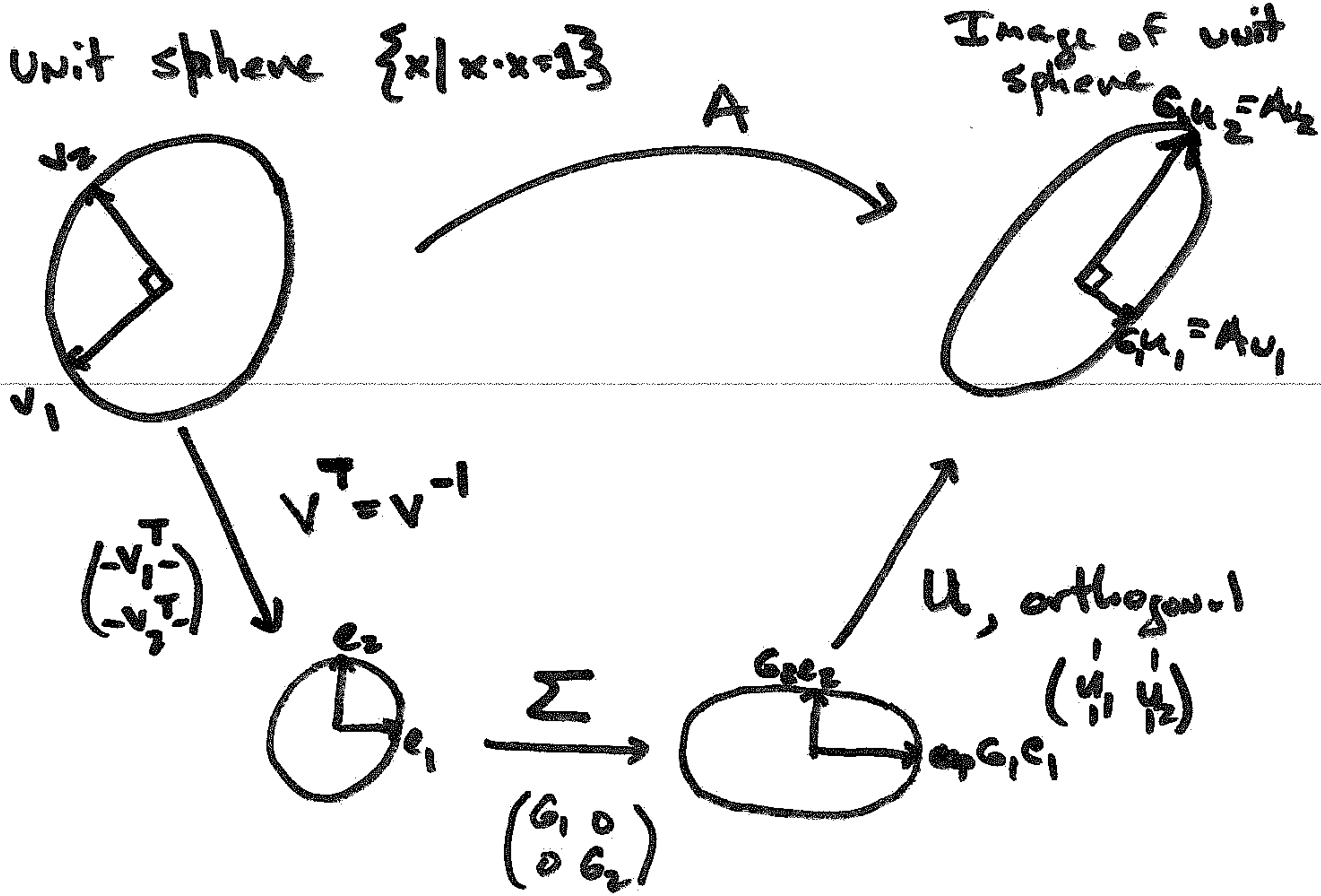
$$\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & \text{\huge 0} & \\ & \text{\huge 0} & \ddots & \\ & & & \sigma_n \end{pmatrix}$$

<u>and</u>

the entries $\sigma_i \geq 0$ and

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0.$$

Conceptualizing the SVD:



$$A = U \Sigma V^T$$

Def: The values

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$$

are called the singular values of A.

The columns of U are called <u>left</u> singular vectors of A

The columns of V are called <u>right</u> singular vectors of A.

How to Find SVD

Want

$$A = U \Sigma V^T$$

$$\Rightarrow A^T = (U \Sigma V^T)^T = V \Sigma^T U^T$$

So symmetric

$$A^T A = (V \Sigma^T U^T)(U \Sigma V^T)$$

U is orthogonal

$$= V \Sigma^T \cancel{(U^T U)} \Sigma V^T$$

$$= V (\Sigma^T \Sigma) V^T.$$

~~eigenvalue~~ diagonalizing $A^T A$ we have:

$$A^T A = V D V^T \qquad D = \Sigma^T \Sigma.$$

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{array}\right]$$

$$\Sigma^T = \left[\begin{array}{ccc} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{array}\right]$$

$$D = \Sigma^T \Sigma = \left[\begin{array}{ccc|c} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_n^2 & \\ \hline & & & 0 \\ & & & & 0 \end{array}\right]$$

Step 1 for finding SVD:

Diagonalize $A^T A$

- V is the orthogonal matrix such that $VDV^T = A^T A$
- $\Sigma = \left(\begin{array}{ccc|c} \sqrt{\lambda_1} & & & \\ & \ddots & & 0 \\ & & \sqrt{\lambda_n} & \end{array}\right)$ square roots of n largest eigenvalues.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Step 1: Compute

$A^T A$ and diagonalize it.

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

1. Find eigenvalues
2. Find orthonormal basis of eigenvectors.

$$p_A(t) = \begin{vmatrix} 1-t & 0 & 1 \\ 0 & 1-t & 1 \\ 1 & 1 & 2-t \end{vmatrix}$$

$$= (1-t) \begin{vmatrix} 1-t & 1 \\ 1 & 2-t \end{vmatrix} +$$

$$\begin{vmatrix} 0 & 1-t \\ 1 & 1 \end{vmatrix}$$

$$= (1-t)^2(2-t) - (1-t)$$

$$- (1-t)$$

Eigenvalues

$\lambda = 1, 0, 3.$

Singular values of A are

$\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{1} = 1$

! Ignore last $m-n$ eigenvalues.

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Find V s.t. $A^TA = VDV^T$

For each λ compute $\cancel{\mathrm{Ker}(A - \lambda I)}$

Ker(A^TA - \lamda I)

$\vdots$

Answer:

$$v_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \lambda_1 = 3, \; \sigma_1 = \sqrt{3}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = 1, \; \sigma_2 = 1$$

$$v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \lambda_3 = 0$$

$$V = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix}.$$

Step 2: Find U.

~~For each~~ For first n singular vects $v_1, \dots, v_n$ compute Av_i

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

! If $n > m$, fill out list by finding basis for $\mathrm{Im}(A)^{\perp}$

$$v_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\|Av_1\|} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{3}\sqrt{6}} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{18}} \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

$$u_2 = \frac{Av_2}{\|Av_2\|} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

Then U is

$$U = \begin{bmatrix} 3/\sqrt{18} & -1/\sqrt{2} \\ 3/\sqrt{18} & 1/\sqrt{2} \end{bmatrix}.$$

So

$$\boxed{A = U \Sigma V^T.}$$

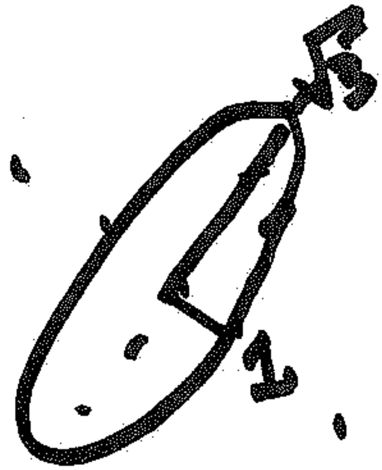
Remark:

$$\{x \cdot x = 1\}$$

x|

$$\xrightarrow{A}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



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