Lecture 32

Quadratic Forms II

Last lecture

① A quadratic form is a polynomial in many variables with each term degree 2 (i.e. $X_i X_j$ or X_i^2)

E.g. $3x^2 + 5xy + 2y^2$

② A quadratic form can be encoded as

$\vec{x} \cdot S\vec{x}$ where S is a symmetric matrix

E.g $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{pmatrix} 3 & 5/2 \\ 5/2 & 2 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

We can analyze quadratic form by analyzing S.

Principal Axes thm: Let $q(x) = x \cdot Sx$ be a quadratic form, let $v_1, \ldots, v_n$ be ~~the~~ a orthonormal eigenbasis for S, then

$$q(x) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \ldots + \lambda_n c_n^2$$

where

$$[x]_{v_1, \ldots, v_n} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\lambda_1, \ldots, \lambda_n$ eigenvalues of S

Today

Problem: Graph

$$\vec{x} \cdot S\vec{x} = K$$

for some fixed # K.

To start, we'll consider the case $S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ a diagonal matrix.

$$\vec{x} \cdot S\vec{x} = \lambda_1 x^2 + \lambda_2 y^2$$

For simplicity assume $\lambda_1 \geq \lambda_2$ and $\lambda_i \neq 0$.

E.g. $S = \begin{pmatrix} 9 & 0 \\ 0 & 2 \end{pmatrix}$ or $S = \begin{pmatrix} 9 & 0 \\ 0 & -13 \end{pmatrix}$

Case 1: $\lambda_1 \geq \lambda_2 > 0$

(positive eigenvalue case)

$\vec{x} \cdot S\vec{x} = \lambda_1 x^2 + \lambda_2 y^2$

e.g. $x \cdot S\vec{x} = 7x^2 + 2y^2$

For $K < 0$

$$x \cdot Sx = K$$

<u>No</u> solutions

(e.g. $7x^2 + 5y^2 = -1$

<u>No</u> solutions)

For $K > 0$, the solution set is an ellipse.

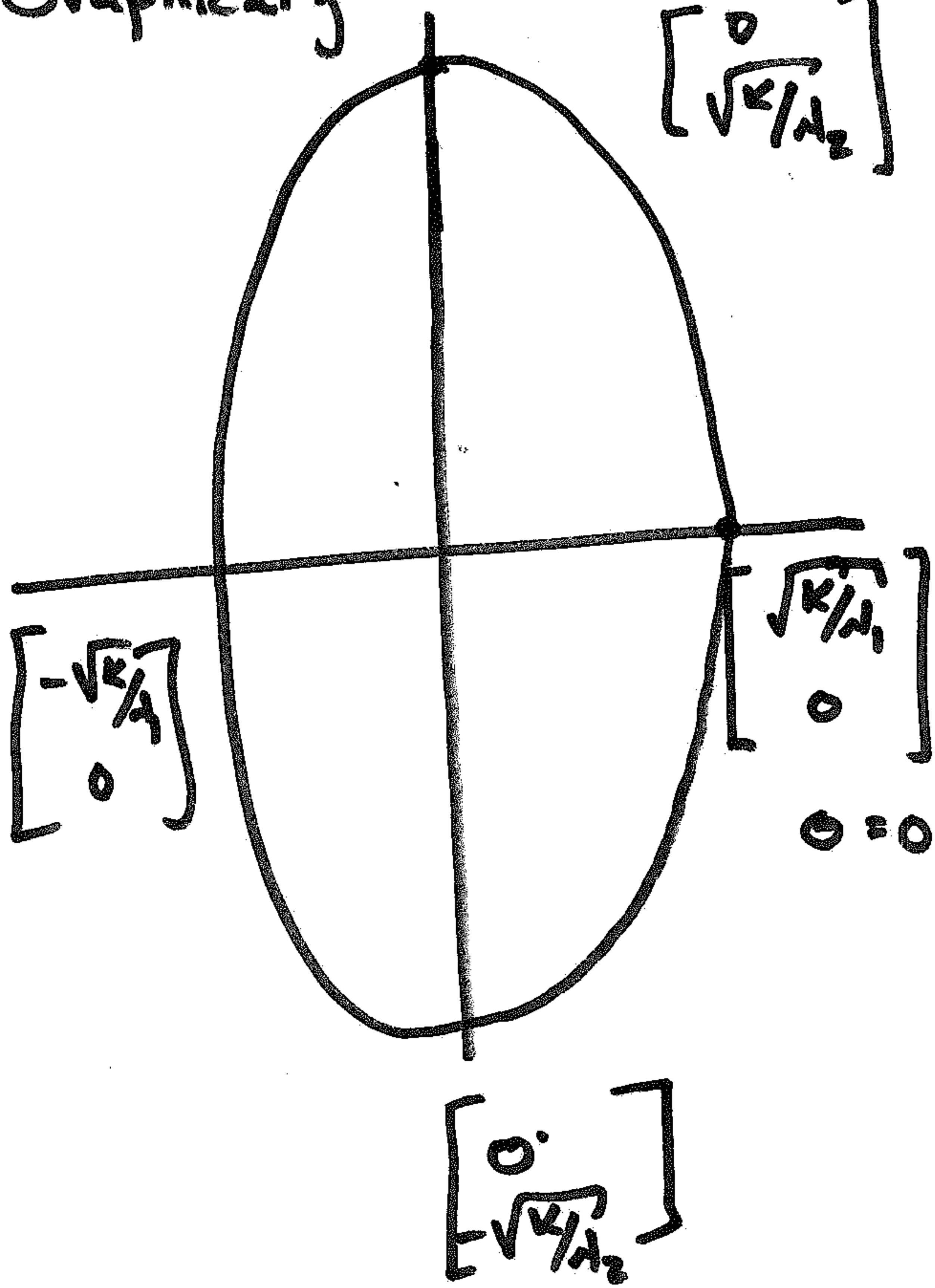
The solution set to

$$\lambda_1 x^2 + \lambda_2 y^2 = K$$

<u>is</u>

$$\left\{ K^{1/2} \begin{bmatrix} \dfrac{\cos\theta}{\lambda_1^{1/2}} \\ \dfrac{\sin\theta}{\lambda_2^{1/2}} \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

Graphically:



Note $\vec{x} \cdot S\vec{x} \geq 0$ and

$\vec{x} \cdot Sx = 0$ if and

only if $x = 0$.

Case 2: $\lambda_1 > 0 > \lambda_2$

One positive eigenvalue + one Negative eigenvalue.

$$S = \begin{bmatrix} 7 & 0 \\ 0 & -2 \end{bmatrix} \qquad \left(S = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \right)$$

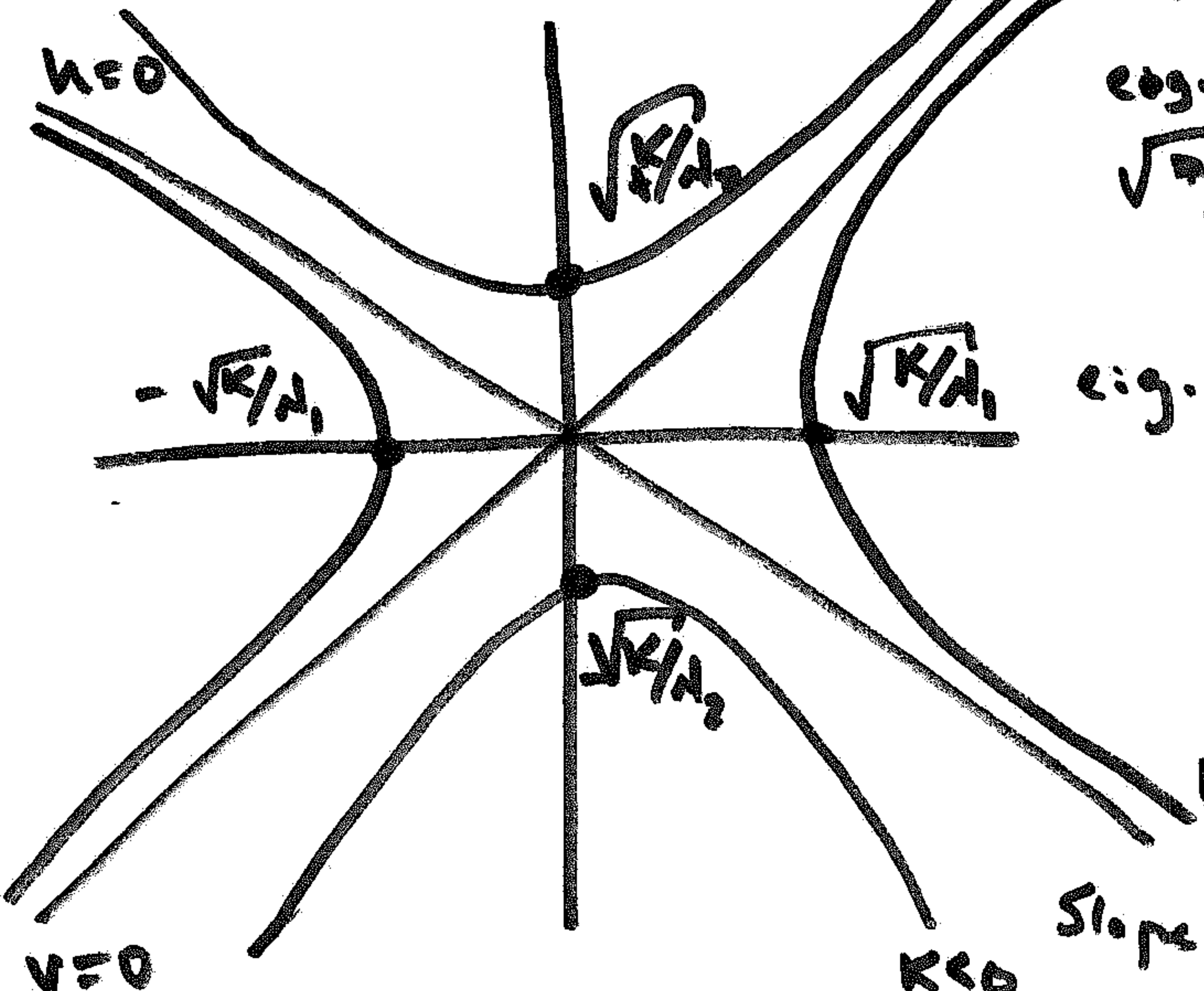
$$\vec{x} \cdot S\vec{x} = 7x^2 - 2y^2$$
$$= (\sqrt{7}x - \sqrt{2}y)(\sqrt{7}x + \sqrt{2}y)$$

Consider $k=0$, $\vec{x} \cdot S\vec{x} = k$

$$\vec{x} \cdot S\vec{x} = 0 \iff \begin{matrix} \sqrt{7}x - \sqrt{2}y = 0 \\ \sqrt{7}x + \sqrt{2}y = 0 \end{matrix}$$

$$\left(\begin{matrix} \sqrt{\lambda_1}x - \sqrt{-\lambda_2}y = 0 \\ \sqrt{\lambda_1}x + \sqrt{-\lambda_2}y = 0 \end{matrix} \right)$$

Graph of
$X \cdot SX = 0.$



Slope $\sqrt{\lambda_1/-\lambda_2}$

e.g. $\sqrt{7/2}$

e.g. $\sqrt{k/2}$

Slope $-\sqrt{\lambda_1/-\lambda_2}$

e.g. $-\sqrt{7/2}$

$$\sqrt{\lambda_1 / -\lambda_2}$$

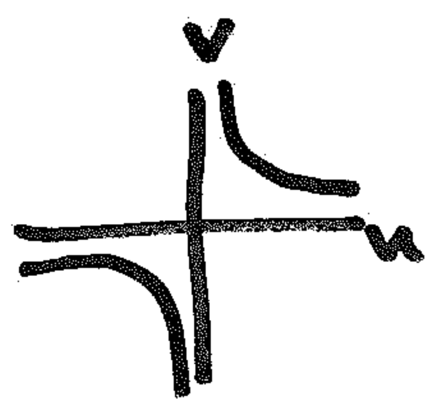
$K > 0$

Still use factorization

$$u = \sqrt{7}\,x - \sqrt{2}\,y$$

$$v = \sqrt{7}\,x + \sqrt{2}\,y$$

$$x \cdot Sx = uv$$



$$uv = K$$

$$v = \frac{K}{u}$$

$$7x^2 - 2y^2 = K \quad \text{if } y = 0$$

$$7x^2 = K \Rightarrow x = \pm\sqrt{K/7}$$

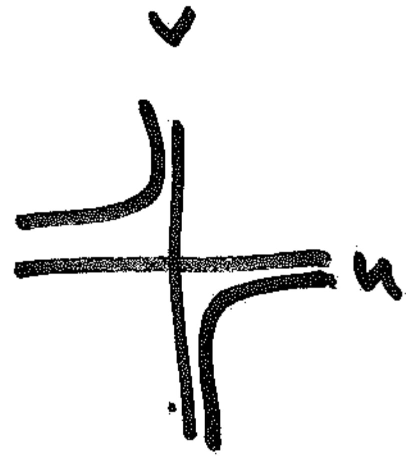
A hyperbola

if $K < 0$

$$7x^2 - 2y^2 = K$$

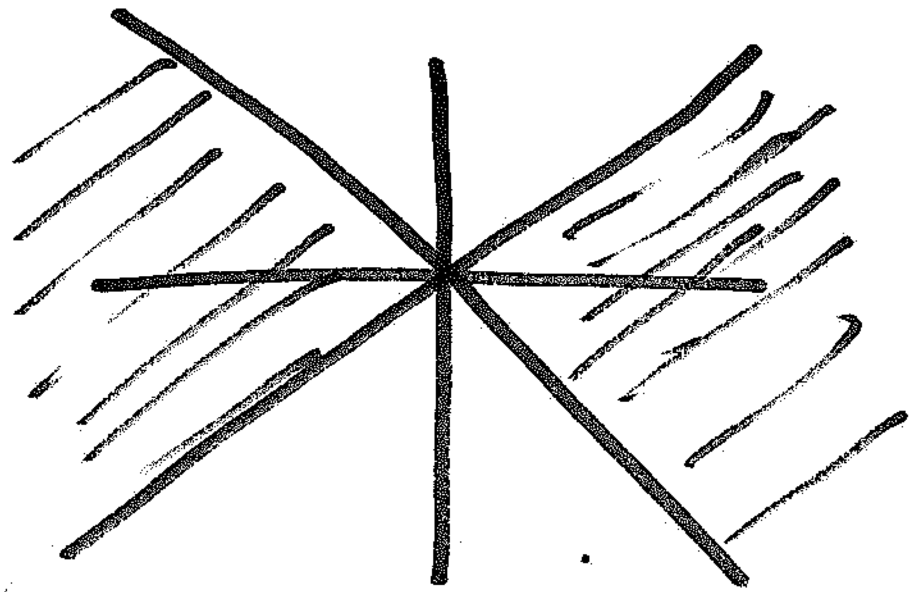
if $x = 0$ then

$$y = \pm\sqrt{K/-2}$$

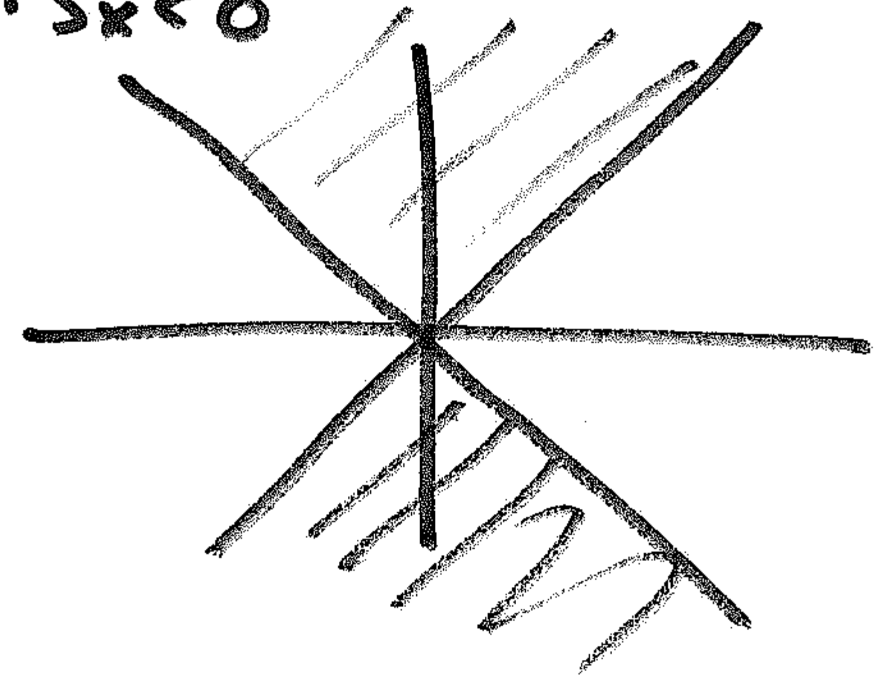


IN particula

$x \cdot Sx > 0$



$x \cdot Sx < 0$



Case 3: $\lambda_1, \lambda_2 < 0$ Negative eigenvalues

Same analysis in 1, but

$x \cdot Sx < 0$ for all K for

each Negative value of

K $x \cdot Sx = k$ is

an ellipse.

Def: A quadratic form is called positive (resp. Negative) definite if $x \cdot Sx > 0$ whenever $x \neq 0$ ($\text{res } x \cdot Sx < 0$ whenever $x \neq 0$)

Case 1: positive definite

Case 3: Negative definite

Thm: A quadratic form

$$x \cdot Sx$$

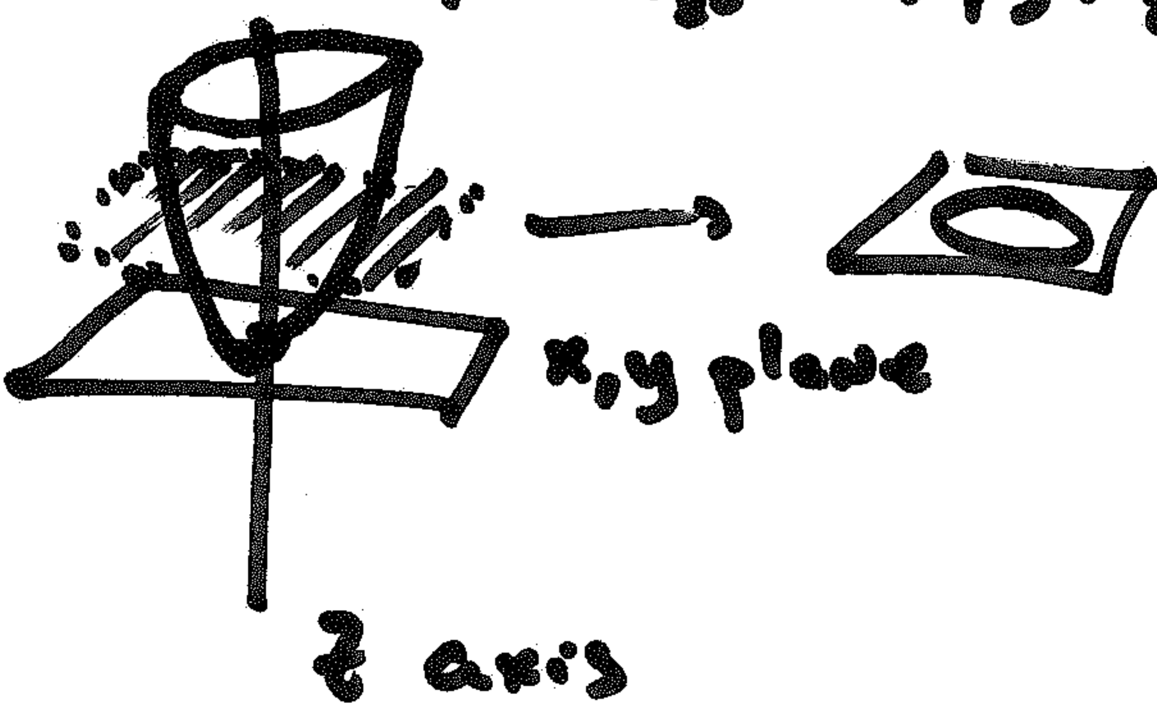
is positive definite if and only if all eigenvalues of S are positive.

Thm: The sets $x \cdot Sx = k$ are ellipses (or their higher dimensional analogs) if and only if $x \cdot Sx$ is positive/ definite
negative

Alt: graph the function

$$x \longmapsto x \cdot Sx$$

$$\lambda_1 x^2 + \lambda_2 y^2 \quad \lambda_1, \lambda_2 > 0$$



$\vec{0}$ is a minimal value of a positive definite quadratic form.

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$\lambda_1, \ldots, \lambda_n$ eigenvalues of S

Example:

Sketch the curve

$$8x^2 - 4xy + 5y^2 = 1.$$

Solution:

Step 1: write quadratic form as $x \cdot Sx$

$$S = \begin{bmatrix} 8 & -4/2 \\ -4/2 & 5 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}.$$

Step 2: Diagonalize S using orthogonal matrix

① $P_A(t) = t^2 - 13t + 44$

$\lambda_1 = 9, \quad \lambda_2 = 4.$

(2) Find eigenspaces

Span $\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$ ev. 9

Span $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ ev. 4

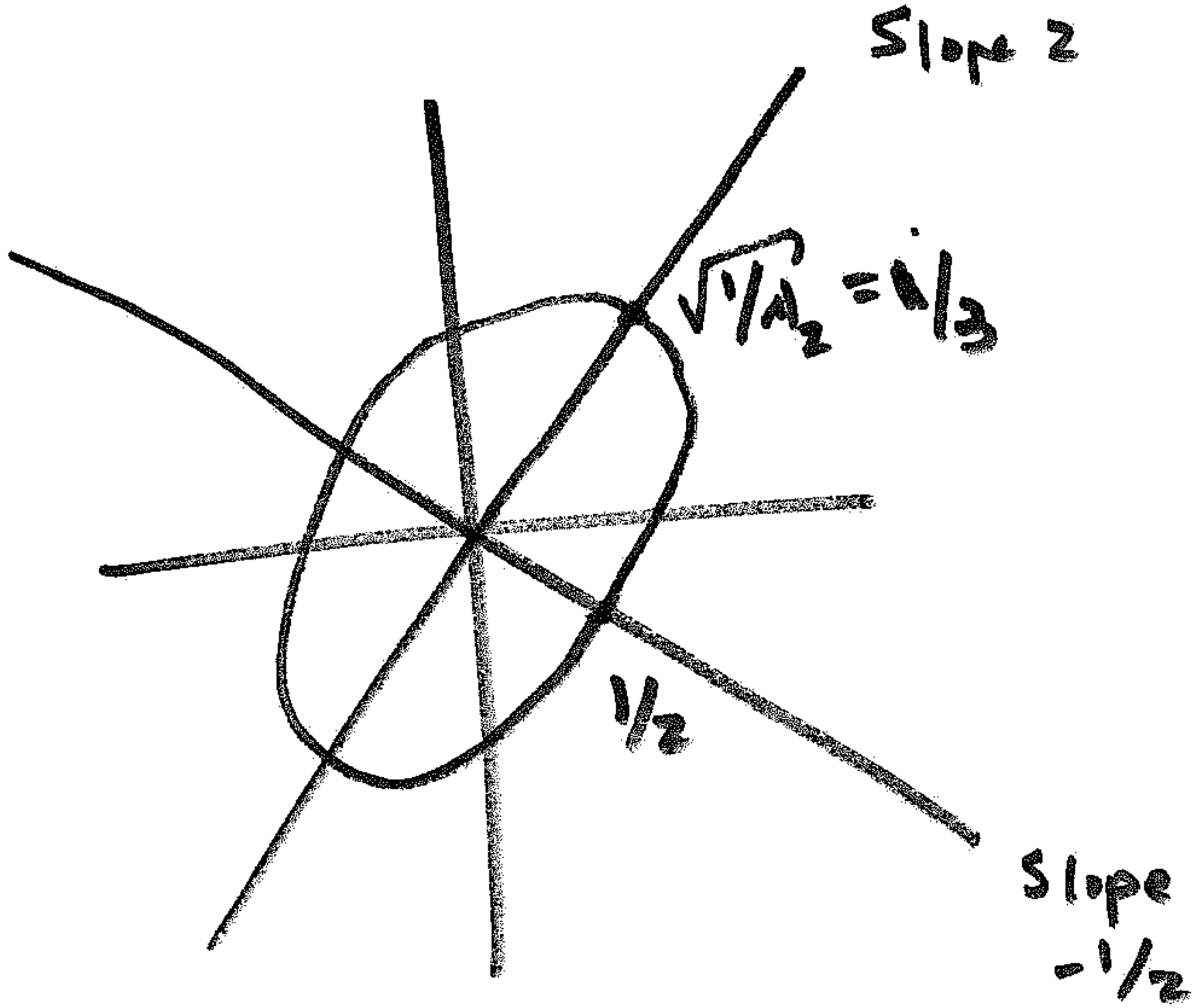
choose unit vects

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

in this coordinate system the form is given by

" $\lambda_1 x^2 + \lambda_2 y^2 = 1$ "

$$4x^2 + 9y^2 = 1.$$


Slope 2
$\sqrt{1/\lambda_2} = 1/3$
1/2
Slope
-1/2