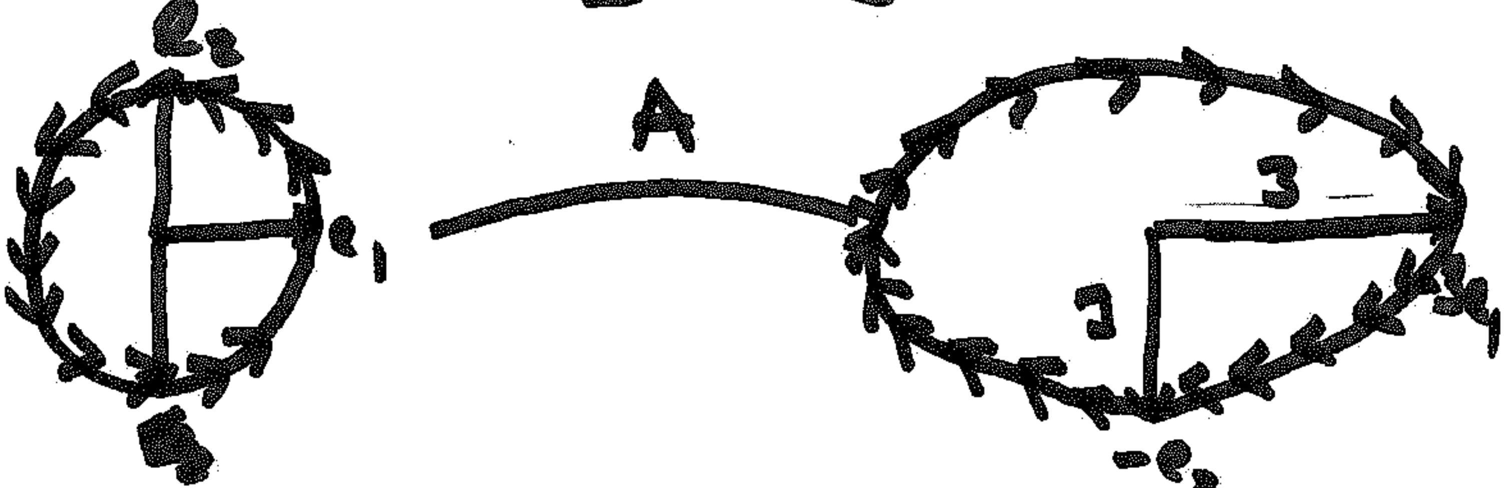
Lecture 31

Quadratic Forms I

Question: what is the image of the unit circle under a linear transformation $A: \mathbb{R}^2 \to \mathbb{R}^2$?

Ex 1: $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ diagonal matrix



Algebraically, there are two ways to describe an ellipse.

① Parametrically:

Circle $\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$.

Resulting ellipse

$$\left\{ A \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 3\cos\theta \\ -\sin\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}.$$

---

② Solution to alg. equation

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 = 1 \right\}$$

Circle

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Ellipse!

$$\left\{ A \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 = 1 \right\}$$

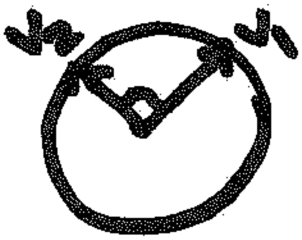
$$\left\{ \begin{bmatrix} 3x \\ -y \end{bmatrix} : x^2 + y^2 = 1 \right\}.$$

$3x = u$, $-y = v \implies x = u/3$, $y = -v$

$$= \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \begin{array}{c} q(u,v) = 1 \\ \| \\ \frac{u^2}{9} + (-v)^2 \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \frac{u^2}{9} + v^2 = 1 \right\}.$$

Ex 2 : $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Symmetric Matrix



$\xrightarrow{A}$??

By spectral thm A has an orthonormal basis of eigenvectors

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{e.v. } \lambda = 3$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{e.v. } \lambda = -1.$$

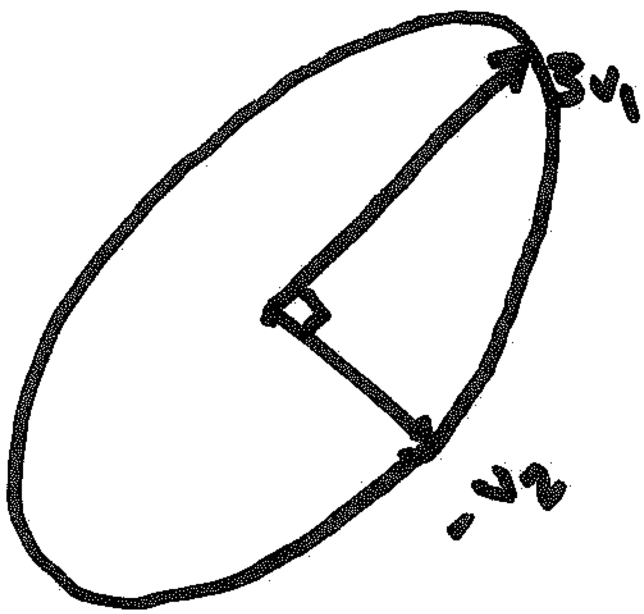
Because $v_1 \perp v_2$ we can parametrize the circle:

$$\{ \cos\theta \, v_1 + \sin\theta \, v_2 : \theta \in \mathbb{R} \}.$$

apply A:

$$\{ A(\cos\theta\, v_1 + \sin\theta\, v_2) : \theta \in \mathbb{R} \}$$

$$= \{ 3\cos\theta\, v_1 + 1\sin\theta\, v_2 : \theta \in \mathbb{R} \}$$



(Parametric description).

(2) Solution set to algebraic equation

Circle

$$\left\{\begin{bmatrix}x\\y\end{bmatrix} : x^2+y^2=1\right\}$$

$$= \{\vec{x} : \vec{x}\cdot\vec{x} = 1\}.$$

$$\xrightarrow{A} \{A\vec{x} \mid \vec{x}\cdot\vec{x} = 1\}.$$

$\vec{w} = \begin{bmatrix}u\\v\end{bmatrix}$

$A\vec{x} = \vec{w}$

$\vec{x} = A^{-1}\vec{w}$

$$= \{\vec{w} \mid (A^{-1}\vec{w})\cdot(A^{-1}\vec{w}) = 1\}.$$

$$= \{\vec{w} \mid \vec{w}\cdot(A^{-1})^T A^{-1}\vec{w} = 1\}.$$

(This calculation did not depend on the fact A is symmetric

Unpacking:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

$$(A^{-1})^T = (A^T)^{-1}.$$

$$(A^{-1})^T A^{-1} = \frac{1}{9}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

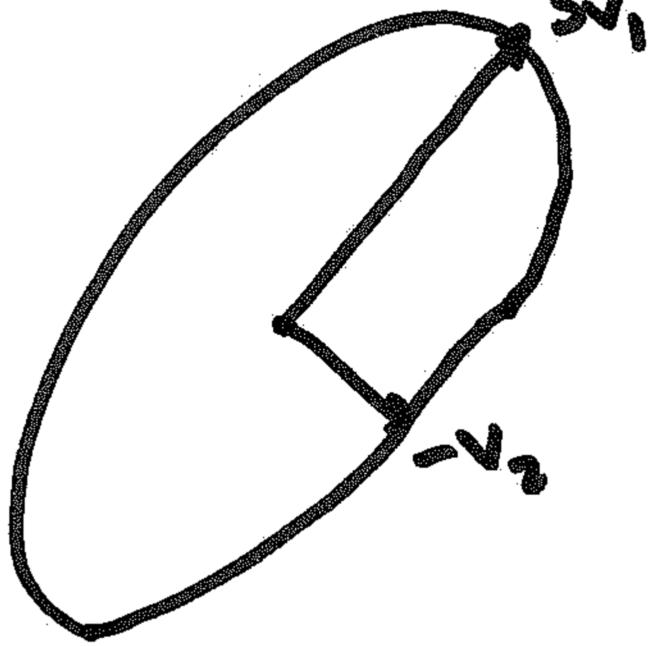
$$\vec{w} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \cdot \left[\left(\frac{1}{9}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}\right)\begin{bmatrix} u \\ v \end{bmatrix}\right].$$

$$\frac{1}{9}\begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} 5u - 4v \\ -4u + 5v \end{bmatrix}$$

$$\frac{1}{9}\Big[(5u-4v)u+(-4u+5v)v\Big]$$

$$=\frac{5}{9}u^2-\frac{8}{9}uv+\frac{5}{9}v^2=1$$



Thm: If A is <u>any</u> invertible linear transformation $A: \mathbb{R}^2 \to \mathbb{R}^2$ the image of the unit circle under A is

$$\left\{ \vec{w} \mid \vec{w} \cdot \left((A^T)^{-1} A^{-1} \vec{w} \right) = 1 \right.$$

. . . . - - - -

$(A^T)^{-1}A^{-1}$ is a symmetric matrix.

☆ We'll study expressions of the form

$\vec{w} \cdot S\vec{w}$ where S is a symmetric matrix.

Ex: $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$\vec{x} \cdot S\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \left(\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} \cdot \left(\begin{bmatrix} ax + by \\ bx + cy \end{bmatrix} \right)$$

$$= x(ax + by) + y(bx + cy)$$

$$= ax^2 + 2bxy + cy^2.$$

Def: A function $q : \mathbb{R}^n \to \mathbb{R}$ is called a quadratic form if there are real numbers $c_{ij} \in \mathbb{R}$ such that

$$q(\vec{x}) = \sum c_{ij} x_i x_j$$

(ie i can equal j)

Examples

① $q : \mathbb{R}^2 \to \mathbb{R}$

$$q(x,y) = x^2 + y^2 + 5xy$$

② $q : \mathbb{R}^3 \to \mathbb{R}$

$$q(x,y,z) = 2x^2 + 3xy + 5z^2 + 4xz + y^2$$

Thm: A quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ can be written uniquely as

$$q(\vec{x}) = \vec{x} \cdot S\vec{x}$$

where S is a symmetric matrix.

Specifically,

$$S = \begin{bmatrix} s_{ij} \end{bmatrix}$$

where

s_{ii} = coefficient of $x_i^2 = c_{ii}$

and

$s_{ij} = \frac{1}{2}$ (coefficient of $x_i x_j$)

$= c_{ij}$

Examples :

① $q = x^2 + y^2 + 5xy$

$$S = \begin{bmatrix} 1 & 5/2 \\ 5/2 & 1 \end{bmatrix}.$$

$$\vec{x} \cdot S\vec{x} = x^2 + y^2 + 5xy.$$

Ex 2:

$$q = 2x^2 + 3xy + 5z^2 + 4xz + y^2$$

$$S = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 2 & 3/2 & 2 \\ 3/2 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix} \end{array}$$

<u>So</u>

$$q(x,y,z) = \vec{x} \cdot S\vec{x}.$$

Objective:

Given a quadratic form $q(\vec{x}) = \vec{x} \cdot S\vec{x}$

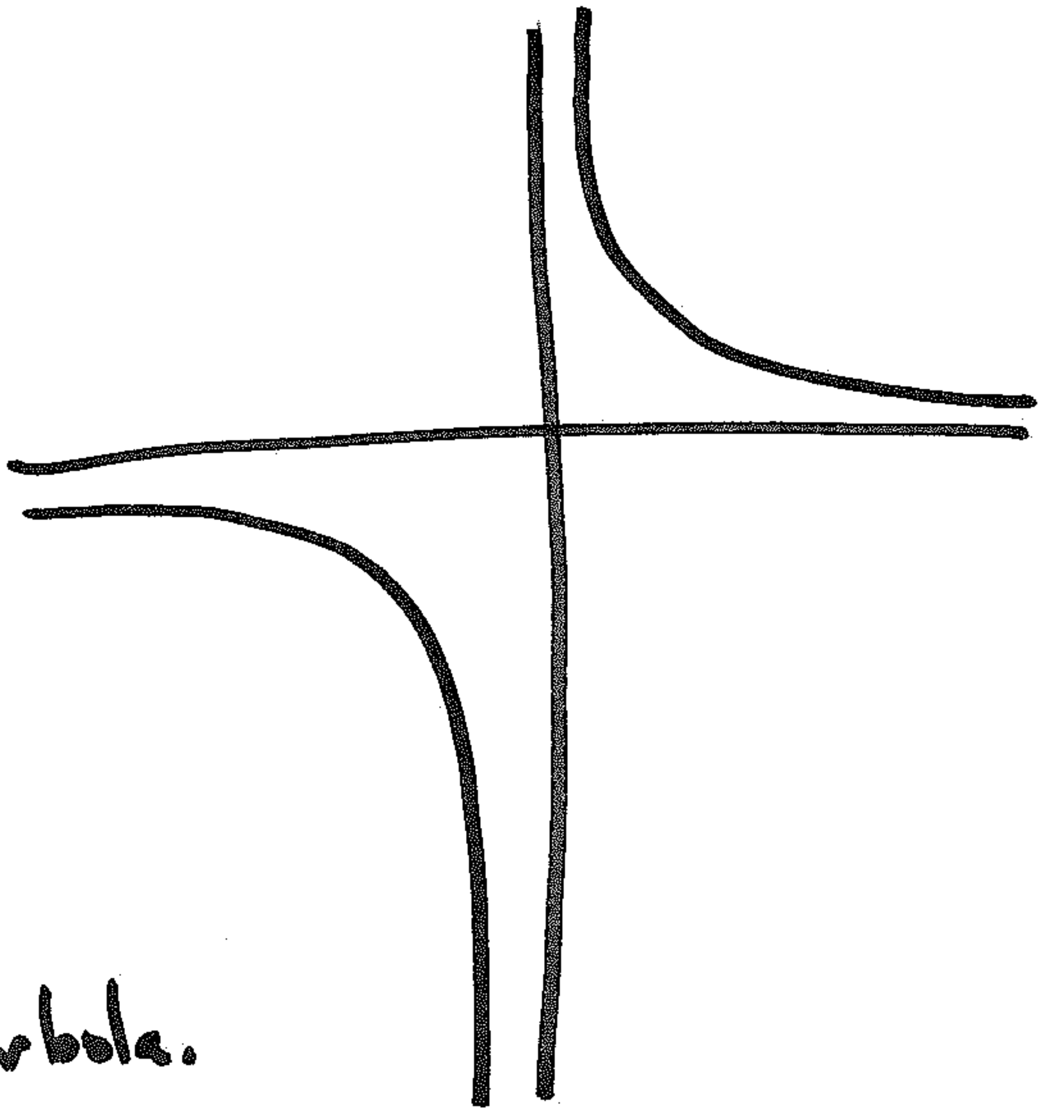
① graph the solution set

② Determine when it's an ellipse.

! Because S is symmetric we can use the spectral thm to diagonalize S and analyze this question. (Next time).

WARNING

$q(\vec{x}) = 1$ is NOT always an ellipse



hyperbola.

$$q(\vec{x}) = xy = 1 \quad \left(y = \frac{1}{x}\right)$$