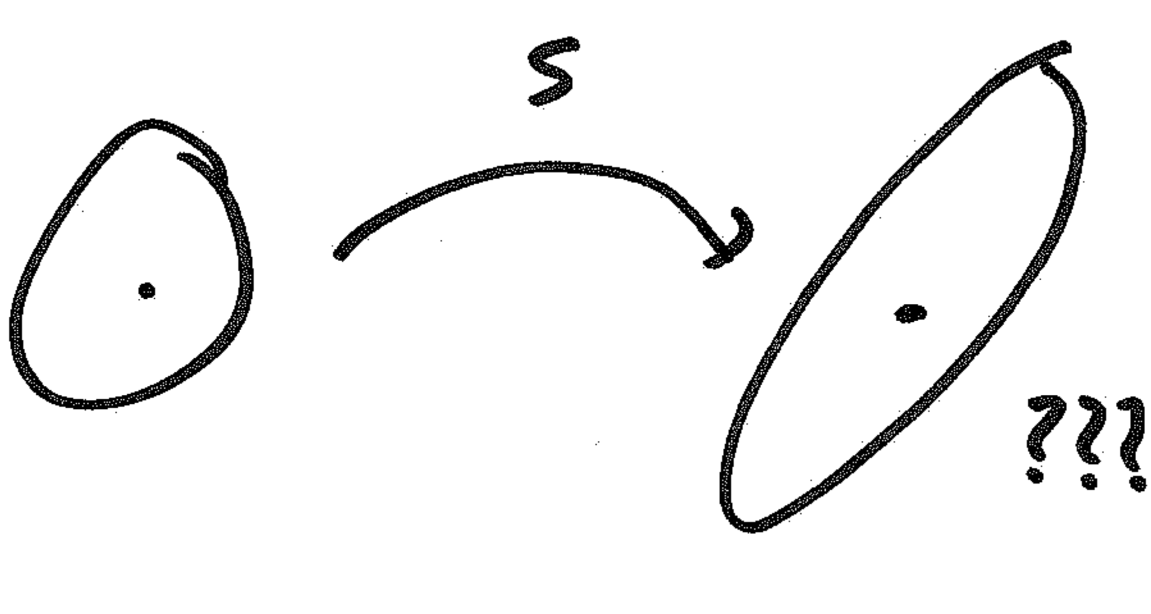
Goal for the next 3 lectures:

Understand which ellipse a circle maps to under a linear transformation.



Lecture 30: Symmetric Matrices.

Def: A matrix A is called symmetric if $A^T = A$.

Examples:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 4 & 1 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

- A is symmetric if its ij-th entry equals its ji-th entry i.e. A is symmetric if it has symmetry across its diagonal.

Main Theorem

Spectral theorem: A matrix A is symmetric if and only if it has an ~~orthonormal~~ orthonormal basis of eigenvectors.

— Equivalently —

A is symmetric if and only if there is a diagonal matrix D and ~~a~~ an orthogonal matrix S such that

$$A = SDS^{-1} = SDS^{T}.$$

Example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

① $p_A(t) = (1-t)^2 - 4$

$= t^2 - 2t - 3.$

$= (t+1)(t-3)$

Eigenvalues $\lambda = -1, 3$.

② Eigenspaces

$\lambda = -1$.

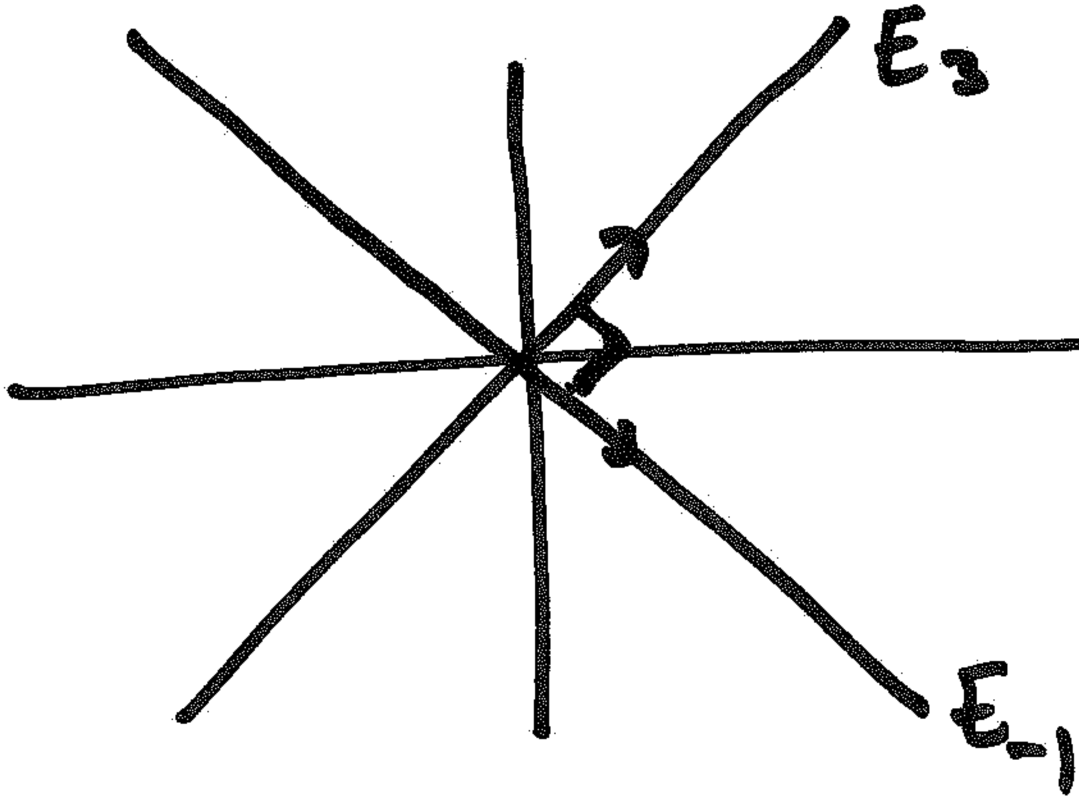
$A - \lambda I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \underset{\text{reduce}}{\overset{\text{row}}{\Longrightarrow}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$E_{-1} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ Line of Slope -1.

$\lambda = 3$

$$A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$ Line of Slope 1



Choose vectors in these eigenspaces of length 1.

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 3$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda = -1$$

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S^{-1} = S^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$A = S\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}S^{-1}.$$

Example 2: Orthogonal projection matrices and reflection matrices are symmetric.

Why? For such matrices I can write down an orthonormal basis of eigenvectors $\xRightarrow{\text{Spectral thm}}$ symmetric.

Example $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ Orthogonal Projection onto line of slope 1.

Facts about transposes:

0. $(A^T)^T = A$.

1. For any A, B such that AB exists

$$(AB)^T = B^T A^T.$$

T reverses order of multiplication

(why? Transpose switches rows and columns, Matrix mult is defined as the dot product of rows + columns, indexing switched when switch rows + columns)

(2) If A is invertible
$$(A^T)^{-1} = (A^{-1})^T.$$

why?

$$AA^{-1} = I$$

$$(A^{-1})^T \cdot A^T = (AA^{-1})^T = I^T = I$$

$$\Rightarrow (A^{-1})^T = (A^T)^{-1}$$

* * 3 If A is an $n \times n$ matrix

$$(Av_1) \cdot v_2 = v_1 \cdot (A^T v_2)$$

Bringing a matrix across a dot product transposes it.

Why?

$$\begin{aligned}(Av_1) \cdot v_2 &= (Av_1)^T v_2 \\ &= v_1^T A^T v_2 \\ &= v_1^T (A^T v_2) \\ &= v_1 \cdot (A^T v_2).\end{aligned}$$

Example 3: For any matrix

$$B,$$

$$BB^T \quad \text{and} \quad B^TB$$

are symmetric.

e.g.

$$B = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \qquad B^TB = \begin{pmatrix} v_1\cdot v_1 & v_2\cdot v_1 \\ v_1\cdot v_2 & v_2\cdot v_2 \end{pmatrix}$$

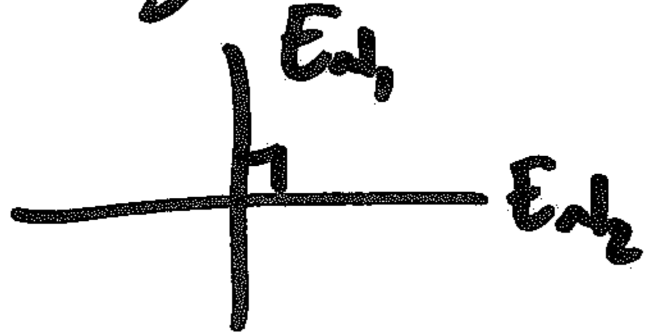
$$B = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \qquad BB^T = \mathrm{Proj}_{\mathrm{Im}(B)}$$

v_1, v_2 orthonormal

Unpacking the Spectral thm

Claim 1: If A is diagonalizable using an orthogonal matrix (orthogonally diagonalizable) then A is symmetric.

Claim 2: If A is symmetric then eigenspaces E_{λ_1}, E_{λ_2} are orthogonal if $\lambda_1 \neq \lambda_2$



Claim 3: All eigenvalues are <u>real</u>.

Claim 4: geo + alg mult are equal for all λ.

Warning: $\dim(E_\lambda)$ can be whatever it wants.

Claim 1: orthogonally diagonalizable $\Rightarrow$ Symmetric

why? $A = SDS^T$

$$
\begin{aligned}
A^T &= (SDS^T)^T \\
&= (S^T)^T D^T S^T \\
&= SDS^T = A. \quad \square
\end{aligned}
$$

Claim 2: Symmetric $\Rightarrow$ perpendicular eigenspaces.

Let v_1, v_2 be eigenvectors with eigenvalues λ_1, λ_2.

$$(Av_1)\cdot v_2 = v_1 \cdot A^T v_2 = v_1 \cdot A v_2$$

$$\begin{array}{ccc} (Av_1)\cdot v_2 & & v_1 \cdot Av_2 \\ \parallel & & \parallel \\ \lambda_1 v_1 \cdot v_2 & & v_1 \cdot \lambda_2 v_2 \\ \parallel & & \parallel \\ \lambda_1 (v_1 \cdot v_2) & & \lambda_2 (v_1 \cdot v_2) \end{array}$$

If $\lambda_1 \neq \lambda_2$ then

$$(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0 \Rightarrow \boxed{v_1 \cdot v_2 = 0.}$$

Claim 3: All eigenvalues are real.

$x+iy$ e.v. with vector $v+iw$

then

$x-iy$ e.v. with vector $v-iw$

Consider

$$A(v+iw)\cdot(v-iw) = (v+iw)\cdot A(v-iw)$$

$$(x+iy)(v+iw)\cdot(v-iw) \qquad (x-iy)(v+iw)\cdot(v-iw)$$

$$(v+iw)\cdot(v-iw) = v\cdot v + w\cdot w > 0$$

<u>so</u> $x+iy = x-iy \Rightarrow y=0$

$\Rightarrow$ e.v. is real

Claim 4: Proof by induction.