Lecture 3: Reduced Row-Echelon Form.

We had been considering the system:

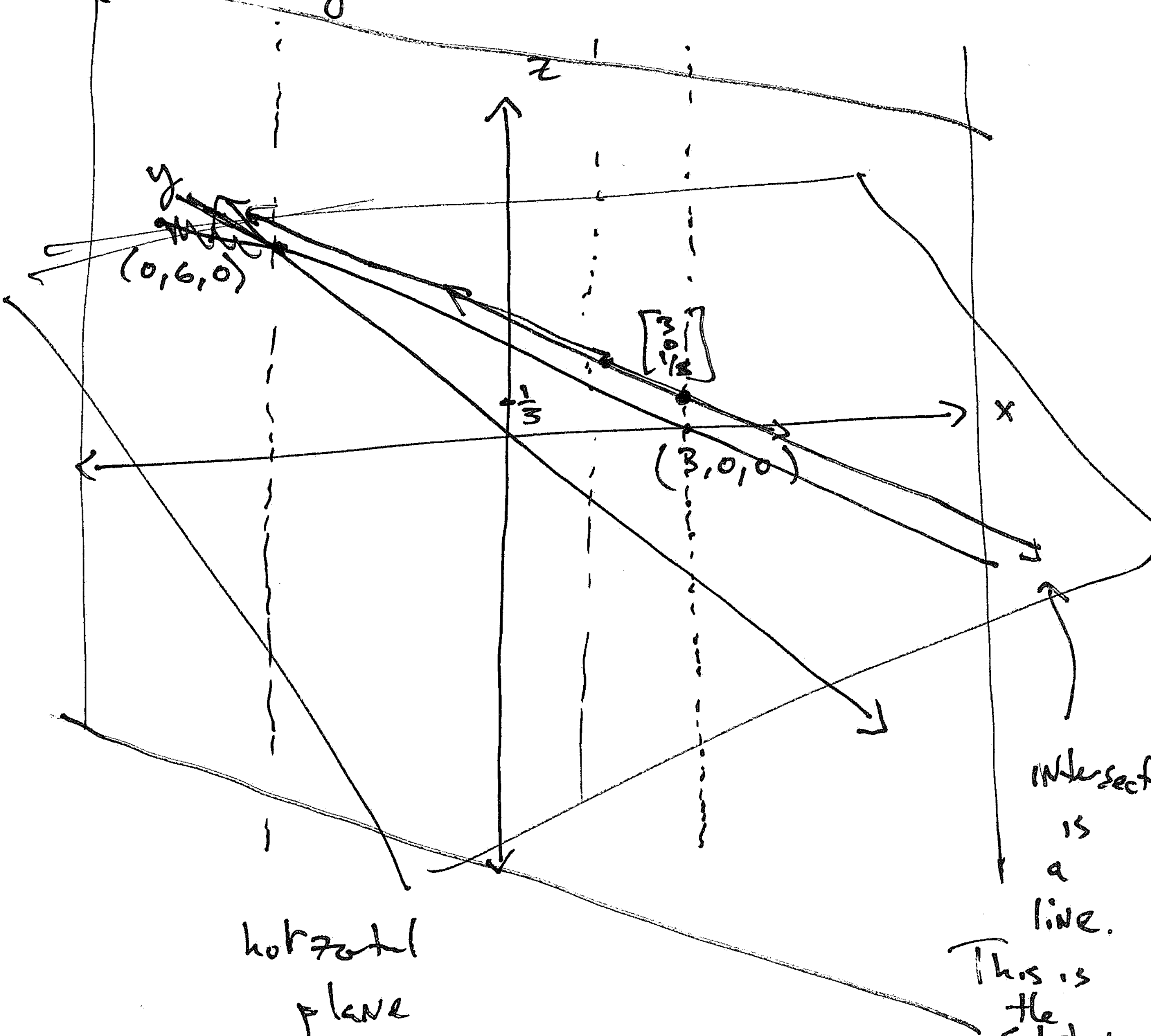
$$9z = 3$$
$$8x + 4y + 3z = 25 \qquad (1)$$
$$2x + y + 6z = 8.$$

We had used <u>elementary row operations</u> on the associated <u>augmented matrix</u> and found that this system has the same solution set as:

$$x + \frac{y}{2} = 3 \qquad (2)$$
$$z = 1/3.$$

What is the solution set?

Geometrically:



horizontal plane

$z = \frac{1}{3}$

intersect is a line.

This is the solution set.

Solutions to

$x + \frac{y}{2} = 3$

Vertical plane

Algebraically:

For every value of y, there is a unique solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - \frac{y}{2} \\ y \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 1/3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \cdot y \\ 1 \cdot y \\ 0 \end{bmatrix}$$

What is nice about the set equations in 2 v.s. 1 is that solutions are determined by the value of y which can be freely ~~chosen~~ and the x and z values of each solution are easily solved for in terms of this y value.

★ Replace each equation with an equation describing how a dependent variable (e.g. x and z) depends on a set of free variables (e.g. y).

What should these equations look like?

P1: The leading coeff of each equation should be 1.

P2: The leading variable in each equation should only appear in that one equation

P3: The leading variables appear in their "natural order" from one equation to the next

(x is in eq. 1
z is in eq. 2)

Translating these properties into ones about an augmented Matrix we have the following definition:

Def : A Matrix is said to be in <u>reduced row-echelon form</u> if

(0) All zero rows occur at the bottom.

(1) The leading <u>non-zero</u> entry of any non-zero row is 1. This entry is called a <u>pivot</u>.

(2) If a column contains a pivot, then all other entries in that column are 0.

(3) If a row contains a pivot, then each row above it contains a pivot further to the left.

~~Examples~~

Which of the following are in row reduced echelon-form?

A. $\left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ No ② does not hold.

B. $\left[\begin{array}{cccc|c} 0 & \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Yes.

C. $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$ No ⓪ does not hold.

D. $\left[\begin{array}{cccc|c} 0 & \boxed{1} & 2 & 3 & 4 \end{array}\right]$. Yes.

A: No B: Yes C: No D: Yes

In A: If you subtract II from III ~~you~~ You obtain a matrix in row reduced echelon form.

In C: you could switch rows 3 and 2 to obtain a matrix in row reduced echelon form

Thm: Given any matrix there is a series of elementary row operations (switches, subtractions + additions, multiplication by non-zero number) ~~such that~~ which puts the matrix in row reduced echelon form.

Example:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \end{array}\right] \xrightarrow{\substack{\mathrm{II}-4\mathrm{I} \\ \mathrm{III}-7\mathrm{I}}} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 1 \end{array}\right]$$

put this matrix in rref using elementary row operations (row reduce the matrix)

$$\xrightarrow{\mathrm{II}/-3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\mathrm{I}-2\mathrm{II} \\ \mathrm{III}+6\mathrm{II}}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

This matrix is in rref.

what is the solution set to the corresponding system?

Third equation says

$$0 = 0 \cdot x + 0 \cdot y + 0 \cdot z = 1$$

So there is <u>no</u> (x, y, z) that satisfies this equation.

The solution set is empty!

Def: A system of linear equations is called consistent if it has a solution.

Thm: A system of linear equations is consistent if and only if

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

does not occur in its reduced row echelon form.