# Lecture 29

Matrix powers w/
complex eigenvalues II.

Warm Up:

Find a (complex) eigenbasis for

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

1. Find char. pol'y

2. Find roots (eigenvalues)

3. For each eigenvalue find eigenspace + a basis.

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1. $(1-t)(4-t)+6$
$$= t^2 - 5t + 10.$$

2. $$\frac{5 \pm \sqrt{25-40}}{2} = \frac{1}{2}\left(5 \pm i\sqrt{15}\right).$$

③ For each $\lambda$ we have to row reduce $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ -3 & 4-\lambda \end{bmatrix}.$$

Rank 2 $\quad \ker(A - \lambda I) = \{0\}$.

No eigenvectors (bad).

Know that there __are__ eigenvectors

Rank < 2.

2nd row some multiple of first.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ -3 & 4-\lambda \end{bmatrix}$$

$$\xRightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1-\lambda & 2 \\ 0 & 0 \end{bmatrix}.$$

$$\operatorname{Ker}(A-\lambda I) = \text{Span}\begin{bmatrix} 2 \\ \lambda-1 \end{bmatrix}.$$

$$\lambda = \frac{1}{2}\left(5 + i\sqrt{15}\right)$$

$$v_1 = 2\begin{bmatrix} 2 \\ \frac{1}{2}\left(3 + i\sqrt{15}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 + i\sqrt{15} \end{bmatrix}$$

eigenvector w/ eigenvalue $\lambda$.

. - - - - - .

$$\lambda = \frac{1}{2}\left(5 - i\sqrt{15}\right)$$

$$v_2 = \begin{bmatrix} 4 \\ 3 - i\sqrt{15} \end{bmatrix}.$$

$v_1, v_2$ eigenbasis.

$\square$

The "actual" lecture began here.

Important thm (from last time):

Two diagonalizable matrices $A$ and $B$ are similar (i.e. $A = SBS^{-1}$ with $S$ real entries) if and only if $A$ and $B$ have the same set of complex eigenvalues (counting multiplicity).

Equivalently:

$$A \sim B \iff P_A(t) = P_B(t)$$

($A$ is similar to $B$)

Warning! Thm is not true w/o diagonalizable assumption

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$p_A(t) = t^2$

$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$p_B(t) = t^2$. $A \not\sim B$

why?

$SBS^{-1} =$ zero matrix $\neq A$.
(for any $S$)

Why is this then true? If

A and B have real eigenvalues

(1) $A = S_1 D S_1^{-1}$

(2) $B = S_2 D S_2^{-1}$

$\Rightarrow D = S_2^{-1} B S_2$

Plug in for D in (1) using (2):

$$A = S_1 ( S_2^{-1} B S_2 ) S_1^{-1}$$

$$= (S_1 S_2^{-1}) B (S_2 S_1^{-1}).$$

If $S = S_1 S_2^{-1}$ then $S^{-1} = S_2 S_1^{-1}$

So

$$A = SBS^{-1} \checkmark$$

If $A$ and $B$ have complex entries there is a choice of $S_1, S_2$ such that $S_1 S_2^{-1}$ has real entries. $\square$.

Today, I'll focus on the 2x2 case of this then and show/find the matrix S such that

$$A = S B S^{-1}$$

when A, B have complex eigenvalues.

# Finding S when A has complex eigenvalues.

Consider:

| | | |
|---|---|---|
| $A$ | $\lambda = x+iy$ | $v+iw$ |
| $2\times 2$ matrix with real entries | complex (non-real) eigenvalue | eigenvector w/ eigenvalue $\lambda$ |

e.g.

| | | |
|---|---|---|
| $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ | $\frac{1}{2}\left(5+i\sqrt{15}\right)$ | $\begin{bmatrix} 4 \\ 3+i\sqrt{15} \end{bmatrix}$ |
| | | $= \begin{bmatrix} 4 \\ 3 \end{bmatrix} + i\begin{bmatrix} 0 \\ \sqrt{15} \end{bmatrix}$ |

Observe:

① $A(\vec{v}+i\vec{w}) = \underbrace{A\vec{v}}_{\substack{\mathbb{R}^2 \\ \text{(real part)}}} + \underbrace{iA\vec{w}}_{\substack{i\mathbb{R}^2 \\ \text{(imag.)}}}$

② $A(\vec{v}+i\vec{w}) = (x+iy)(\vec{v}+i\vec{w})$

$= x\vec{v} - y\vec{w} + iy\vec{v} + ix\vec{w}.$

$= \underbrace{(x\vec{v} - y\vec{w})}_{\text{real}} + \underbrace{i(y\vec{v}+x\vec{w})}_{\text{imaginary.}}$

Equating real ~~+~~ and imaginary parts we get a pair of equations.

$$\begin{pmatrix}\text{real}\\ \text{part}\end{pmatrix}\quad A\vec{v} = x\vec{v} - y\vec{w}$$

$$\begin{pmatrix}\text{im.}\\ \text{part}\end{pmatrix}\quad A\vec{w} = y\vec{v} + x\vec{w}.$$

So, in $(\vec{w}, \vec{v}) = \beta$ coordinates

$$[A]_{\beta} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

$$\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \quad = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

__Point__: $A$ in $(\vec{w}, \vec{v})$-coordinates is a rotation + scaling matrix.

Equivalently:

if $S = \begin{bmatrix} | & | \\ w & v \\ | & | \end{bmatrix}$ then

$$A = S \begin{pmatrix} x & -y \\ y & x \end{pmatrix} S^{-1}.$$

Thm: Let $A$ be a $2 \times 2$ matrix with eigenvalues $x \pm iy$. If $v + iw$ is an eigenvector with eigenvalue $x + iy$ then

$$A = \begin{bmatrix} | & | \\ w & v \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} x & -y \\ y & x \end{bmatrix}}_{\text{(rotation + scaling)}} \begin{bmatrix} | & | \\ w & v \\ | & | \end{bmatrix}^{-1}.$$

Problem: Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$.

Find a matrix $S$ and a rotation and scaling matrix

$$B = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

such that $A = SBS^{-1}$.

Solution:

- $B$ is determined by a <u>choice</u> of one of the eigenvalues of $A$. Choose one.

(I always choose the one with $y \geq 0$).

$$\frac{1}{2}\left(5+i\sqrt{15}\right) = \lambda = x+iy$$

$$= r\left(\cos\theta + i\sin\theta\right).$$

$$r = \sqrt{x^2+y^2}$$

$$= \sqrt{\frac{25}{4}+\frac{15}{4}}$$

$$= \sqrt{10}.$$

Remark: Another way to find

$$r = \sqrt{\det(A)} = \sqrt{10}.$$

$$\det(A) = \det(B) = \det(rI).$$

$\det(\text{rotation})$

$1 \quad = r^2$

$$\theta = \operatorname{Arccos}\left(x/r\right)$$

$$= \operatorname{Arccos}\left(5/2\sqrt{10}\right)$$

$$\approx 37.76^\circ$$

$$B = \sqrt{10}\begin{bmatrix} 5/2\sqrt{10} & -\sqrt{15}/2\sqrt{10} \\ \sqrt{15}/2\sqrt{10} & 5/2\sqrt{10} \end{bmatrix}.$$

$$\approx \sqrt{10}\begin{bmatrix} \cos(38) & -\sin(38) \\ \sin(38) & \cos(38) \end{bmatrix}.$$

$\lambda = x + iy$ ~~ein~~

eigenvector $v + iw = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ \sqrt{15} \end{bmatrix}$

$$S = \begin{bmatrix} | & | \\ w & v \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ \sqrt{15} & 3 \end{bmatrix}.$$

$$A = SBS^{-1}. \quad \square.$$

Exercise: Do vectors spiral clockwise or counterclockwise under $A$?