Lecture 28

Matrix Powers and complex eigenvalues.
Rotation matrix:

\[ A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

(eigenvalues are \( \cos \theta \pm i \sin \theta \))

Powers of \( A \):

\[ A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \]

(eigenvalues are \( \cos(n\theta) \pm i \sin(n\theta) \))
Under powers of $A$, a non-zero vector $x$ orbits $O$ along a circle of radius $|x^0|$. 

\[ \text{(circle)} \]

i.e. the lengths of iterates $|x|, |Ax|, |A^2x|, \ldots$ are constant.
Def: A matrix of the form

\[
A = r \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

for \( r \in \mathbb{R} \), \( r > 0 \) is called a rotation + scaling matrix.

Powers of \( A \):

\[
A^n = r^n \begin{bmatrix}
\cos(n\theta) & -\sin(n\theta) \\
\sin(n\theta) & \cos(n\theta)
\end{bmatrix}
\]

(aside: eigenvalues of \( A \)

\[ r \( \cos \theta \pm i\sin \theta \). \]
If $x^2$ is a non-zero vector:

The sequence of lengths $1, |(Ax)_1, |A^2x|, \ldots$ grows/shrinks exponentially with rate $r$ if $r > 1$ or if $r < 1$, respectively.

The angles of $x, Ax, A^2x, \ldots$ in polar coordinates orbits around the circle.
Putting these together:

$x$ travels along a circular spiral

\[(r > 1, \ 0 < \theta < \pi)\]

Explicitly, the spiral is

\[\sum 1 \times 1 \times (\frac{x}{1x1} + \left[ \cos \theta \right]), \ \text{for } \theta \in \mathbb{R}^2\]
Warning! There are two orientations for spirals.

- Growth rate is counterclockwise.

\[(r>1, \quad 0<\theta<\pi)\]

\[(r<1, \quad 0>\theta>-\pi)\]
\[ n > 1, \quad 0 < \theta > -\pi \]
\[ \left( r < 1, \quad 0 \leq \theta \leq \pi \right) \]
Geometry of matrices
that are similar to
rotation + scaling matrices:

\[
\text{Rotation } A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

\[
B = SAS^{-1}.
\]
What happens to a circle under a linear transformation $S$?

Square \[\xrightarrow{S}\] Parallelogram

Circle \[\xrightarrow{S}\] Ellipse
Fact: A circle maps to an ellipse under $S$.

If $B$ is similar to rotation & scaling

$x, Bx, B^2x, \ldots$

spiral $\&$ out $/$ in along an elliptical spiral.
Similar to Rotation + Scaling:

- Powers travel along elliptical spiral
- Travel along circular spiral

\[ r = a + b \]

\[ \theta = c \cdot \theta \]
When are two matrices similar? (When is a matrix similar to a rotation + scaling?)

Theorem: Two (complex) diagonalizable matrices are similar (over $\mathbb{R}$) if and only if they have the same set (complex) of eigenvalues.
The eigenvalues of $A = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ are $r (\cos \theta \pm i \sin \theta)$.

If $B$ has complex eigenvalues $r (\cos \theta \pm i \sin \theta)$ then $B$ is similar to the rotation $+ \text{scaling}$ defined by $A$. 
Thm: If $A$ is a $2 \times 2$ matrix with complex eigenvalues $\lambda = r(\cos \theta \pm i \sin \theta)$, then

$$A = r S \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} S^{-1}$$

for some $S$ ($2 \times 2$ over $\mathbb{R}$). Under powers of $A$, a vector $x$:

@ Spiral away from $0$ along an elliptical spiral, i.e. $\|Ax\| > 1$. 
2. Orbit \( o \) along if \( r = 1 \) an ellipse
   i.e. \( |N| = 1 \)

3. Spiral toward \( o \) along an elliptical spiral
   i.e. \( |N| < 1 \).
Example: $A = \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}$

eigenvalues

$\lambda = 1 \pm 4i$

$\approx \sqrt{17} (\cos 76^\circ \pm i \sin 76^\circ)$

$|\lambda| = \sqrt{17} > 1$ so

for any vector $x$, $Ax$, $A^2x$, ...

spirals away from 0.
Warning!

\[ A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

eigenvalues: \( \cos \theta \pm i \sin \theta \)

Same eigenvalues as

\[ B = \begin{pmatrix} \cos (-\theta) & -\sin (-\theta) \\ \sin (-\theta) & \cos (-\theta) \end{pmatrix} \]

Same eigenvalues so similar

\[ S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

reflect x-axis \( S A S^{-1} \neq B \).
For a matrix $A$ with complex eigenvalues

$$\lambda = \nu (\cos \theta \pm i \sin \theta),$$

the growth/decay rate of the lengths

$$|x\rangle, |Ax\rangle, \ldots$$

depends on $\nu = |\lambda|$. The "speed" of the orbit of $x, Ax, A^2x, \ldots$

depend on $\pm \theta = \pm \text{Arg}(\lambda)$.

These depend on eigenvalues, i.e. $A$ up to similarity.
The specific ellipse and the orientation (clockwise v. counterclockwise) depend on the matrix \( A \), i.e. depends on \( A \) (not \( A \) up to similarity).