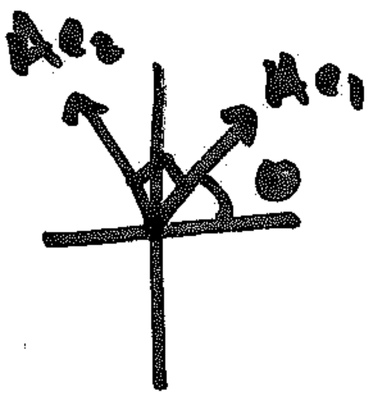
Lecture 27

Complex Eigenvalues

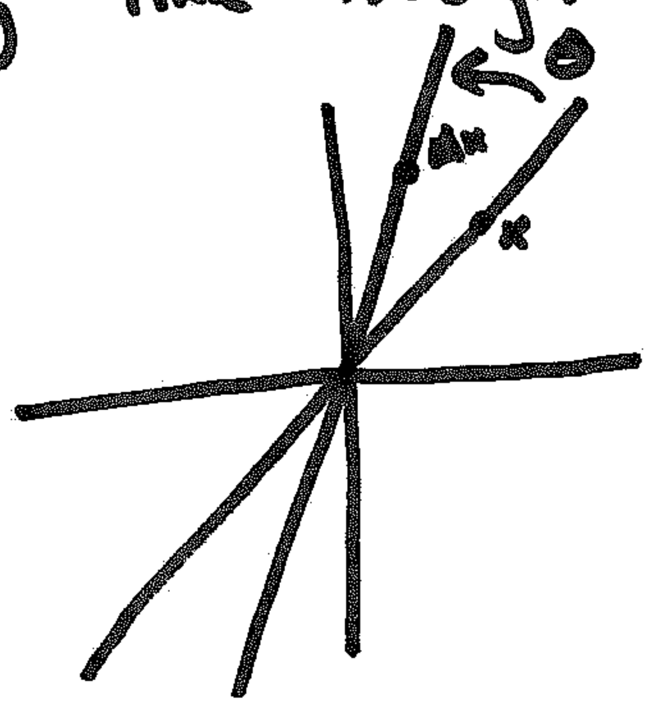
(+ eigenvectors)

Consider the rotation:



$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Geometric Fact: If $\theta \neq 180^\circ, 0^\circ$, then A does not preserve any line through $\vec{0}$.



Algebraically, this implies

$$Av \neq \lambda v$$

for any $v \neq \vec{0}$.

i.e. A does not have any eigenvectors (or eigenvalues).

$$\begin{vmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{vmatrix} =$$

$$(\cos\theta - t)^2 + \sin^2\theta =$$

$$= t^2 - 2\cos\theta \, t + (\cos^2\theta + \sin^2\theta)$$

$$= t^2 - 2\cos\theta \, t + 1.$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\sqrt{4\cos^2\theta - 4} = 2\sqrt{\cos^2\theta - 1}$$

$$= 2\sqrt{-\sin^2\theta}$$

$$= \pm 2\sin^2\theta\sqrt{-1}$$

$$\lambda = \cancel{2}\cos\theta \pm \sin\theta\sqrt{-1}$$

so when $\theta = 0^\circ, 180^\circ$

$\sin\theta\sqrt{-1} \neq 0.$

No real eigenvalues.

Def: Let A be an $n \times n$-matrix

a (complex) eigenvalue of A

is a root λ of

$$P_A(t) = \mathrm{Det}(A - tI),$$

in the complex numbers.

e.g. $\cos\theta \pm i\sin\theta$ are

complex eigenvalues of

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Complex numbers crash course.

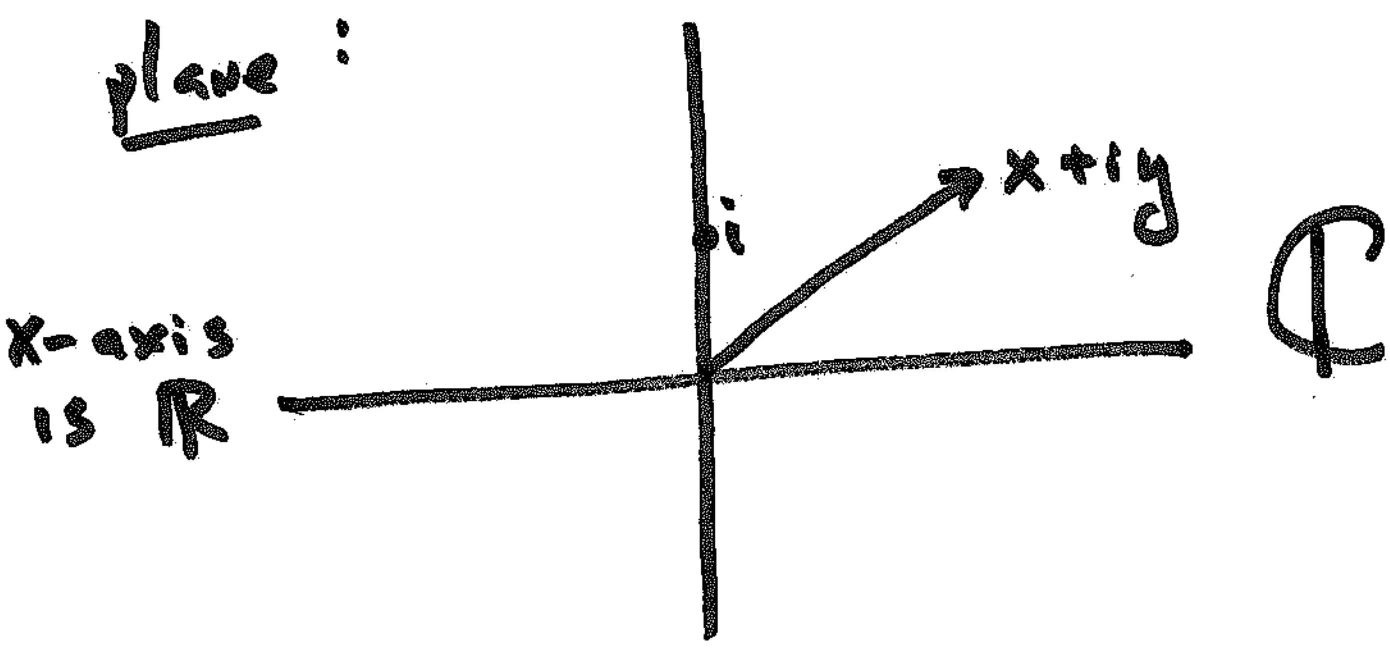
What are complex numbers?

Expressions of the form

$$x+iy$$

$x, y \in \mathbb{R}$, " $i = \sqrt{-1}$. "

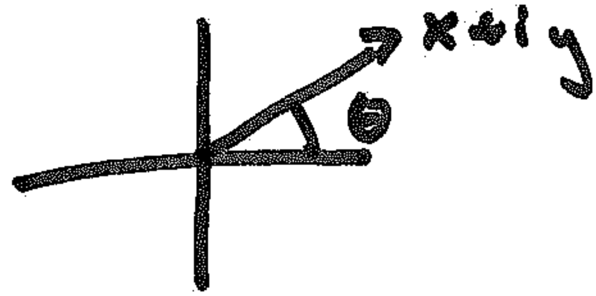
They can be represented as vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in the complex plane:



The length or absolute value of a complex number is

$$|x+iy| = \sqrt{x^2+y^2}$$

We can express a complex # in polar form:



$$x+iy = |x+iy|(\cos\theta + i\sin\theta).$$

$\theta = \operatorname{Arctan}(y/x)$ is

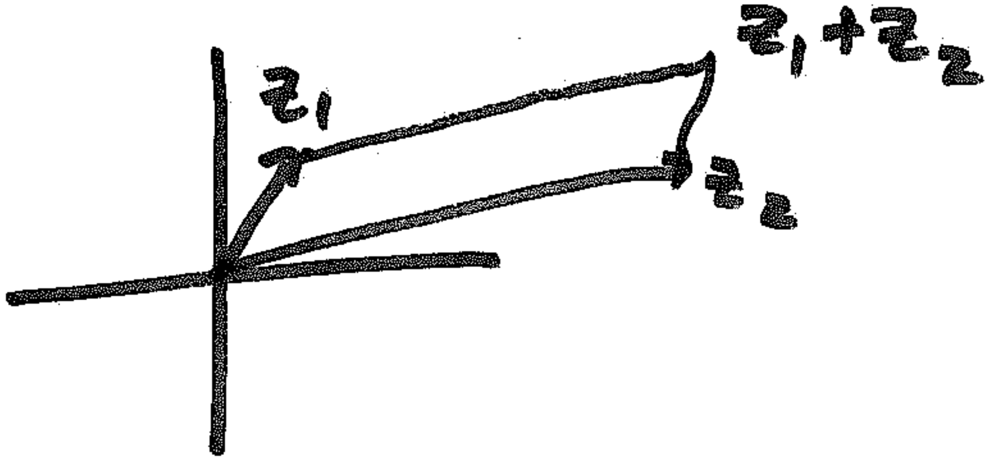
called the <u>argument</u>

of $x+iy$.

You can add complex numbers

$$(x_1 + iy_1) + (x_2 + iy_2)$$

$$= (x_1 + x_2) + i(y_1 + y_2)$$



You can multiply complex #'s!

Use rule $i^2 = -1$.

$$(x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1 x_2 + iy_1 x_2 + iy_2 x_1 + iy_1 iy_2$$

$$= (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + y_2 x_1)$$

IN polar form:

$$r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

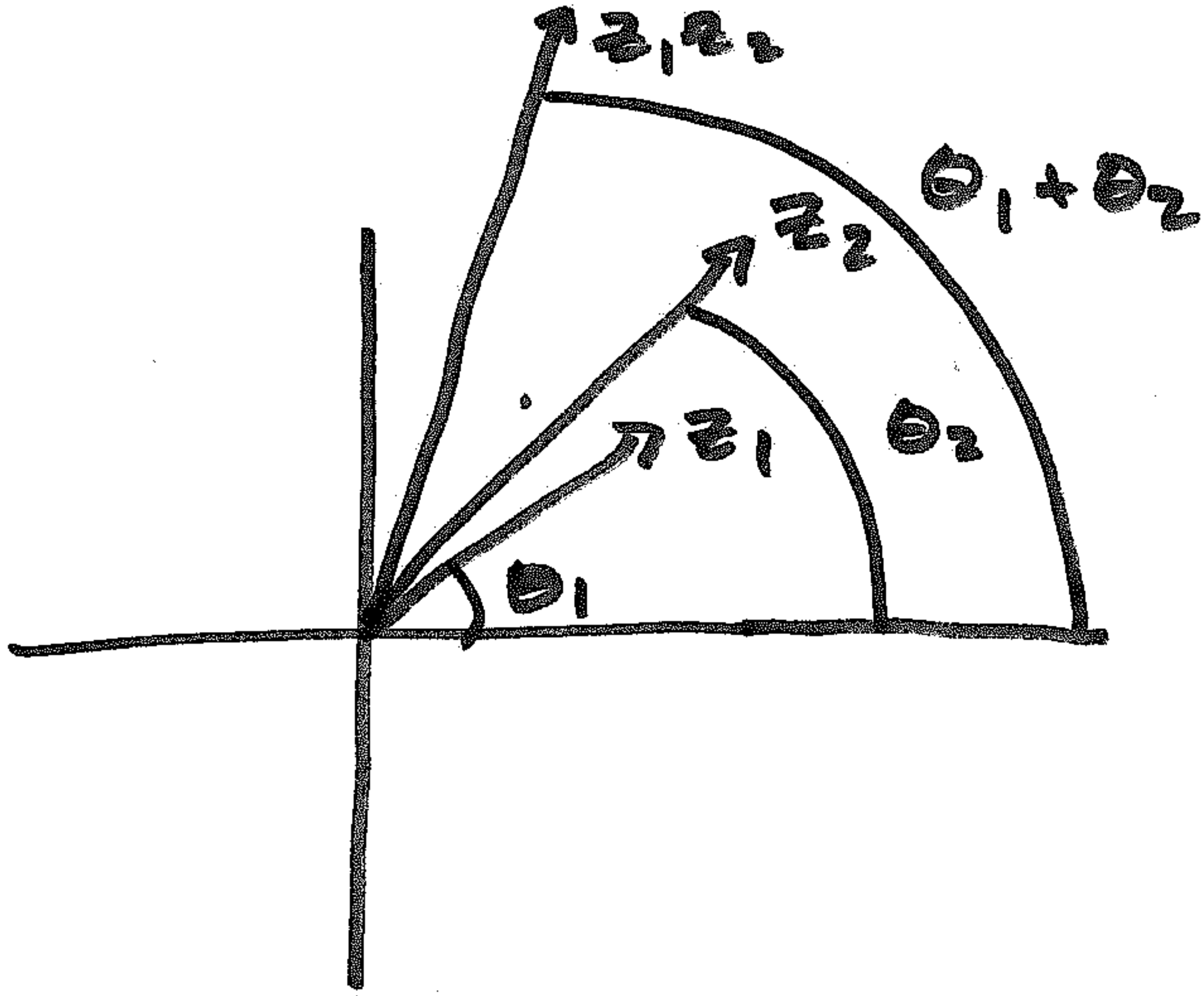
$$= r_1 r_2\big([\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2]$$

$$+ i[\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1]\big)$$

law of sines + cosines

$$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

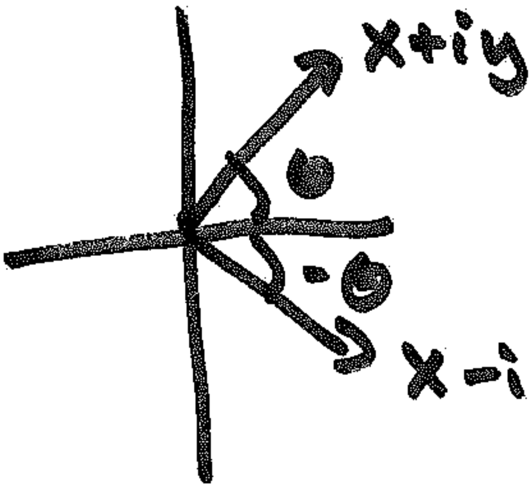
i.e. lengths multiply + Arguments add.



When I multiply by z_2 vectors scale by $|z_2|$ and rotate by $\text{Arg}(z_2)$.

If $x+iy \neq 0$, you can divide.

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$



x[illegible]iy

$x-iy$ is called the complex conjugate of $x+iy$

Fundamental Thm of algebra

Any polynomial

$$f(t) = t^n + a_{n-1} t^{n-1} + \dots + a_0$$

has n roots (counted with multiplicity) in $\mathbb{C}$, i.e.

$$f(t) = (t-\alpha_1)(t-\alpha_2)\dots(t-\alpha_n)$$

$\alpha_1, \dots, \alpha_n \in \mathbb{C}$.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

For us: An $n \times n$ matrix has n complex eigenvalues (counted with multiplicity).

If we allow our matrices to have <u>complex</u> rather than real entries, we can diagonalize <u>more</u> matrices.

Problem: Diagonalize $\begin{bmatrix} 1 & -2 \\ 8 & 1 \end{bmatrix}$

over $\mathbb{C}$.

1. Find eigenvalues.

   1A Compute char poly

$$\begin{vmatrix} 1-t & -2 \\ 8 & 1-t \end{vmatrix} = (1-t)^2 + 16$$

$$= t^2 - 2t + 17$$

   1B Find roots

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 17}}{2}$$

$$\sqrt{4-4\cdot 17} = 2\sqrt{1-17}$$
$$= 2\sqrt{-16}$$
$$= 8\sqrt{-1} = 8i$$

$$\lambda = 1 \pm 4i.$$

(2) Find eigenvectors.

$\lambda = 1+4i$

$$A - \lambda I = \begin{bmatrix} -4i & -2 \\ 8 & -4i \end{bmatrix}.$$

Row reduce to find kernel.

$\lambda = 1 + 4i$

$$\begin{bmatrix} -4i & -2 \\ 8 & -4i \end{bmatrix} \xrightarrow{\mathrm{I}/-4i} \begin{bmatrix} 1 & -i/2 \\ 8 & -4i \end{bmatrix}$$

$$\xrightarrow{\mathrm{II}-8\mathrm{I}} \begin{bmatrix} 1 & -i/2 \\ 0 & 0 \end{bmatrix}$$

$$\frac{1}{-4i} = \frac{1}{-4}\left(\frac{1}{i}\right) = \frac{1}{-4}\left(\frac{-i}{1}\right) = i/4.$$

$$x - \frac{i}{2}y = 0$$

$$\mathrm{Ker}(A - \lambda I) = \mathrm{Span}\left(\begin{bmatrix} i/2 \\ 1 \end{bmatrix}\right).$$

$v_1 = \begin{bmatrix} i/2 \\ 1 \end{bmatrix}$ is a basis.

$\lambda = 1 - 4i$

$$A - \lambda I = \begin{bmatrix} 4i & -2 \\ 8 & 4i \end{bmatrix} \xrightarrow{I/4i} \begin{bmatrix} 1 & i/2 \\ 8 & -4i \end{bmatrix}$$

$$\xrightarrow{II - 8I} \begin{bmatrix} 1 & i/2 \\ 0 & 0 \end{bmatrix}$$

$$\ker(A - \lambda I) = \operatorname{Span}\left(\begin{bmatrix} -i/2 \\ 1 \end{bmatrix} \right).$$

$$v_2 = \begin{bmatrix} -i/2 \\ 1 \end{bmatrix}.$$

Check

$$\begin{bmatrix} 1 & -2 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} i/2 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} i/2 - 2 \\ 4i + 1 \end{bmatrix}$$

$$= (4i+1) \begin{bmatrix} i/2 \\ 1 \end{bmatrix}.$$

eigenvalue. eigenvector

(3) Form D and S

$$D = \begin{bmatrix} 1+4i & 0 \\ 0 & 1-4i \end{bmatrix}$$

$$\begin{array}{cc} S = \begin{bmatrix} \frac{1}{2}i & -\frac{1}{2}i \\ 1 & 1 \end{bmatrix}. \\ \qquad\quad v_1 \qquad\quad v_2 \end{array}$$

S^{-1} same formula

$A = SDS^{-1}$.

Thm: A matrix A is diagonalizable over $\mathbb{C}$ if $p_A(t)$ has n distinct roots.

Why? Every eigenvalue has at least 1 eigenvector.

Cons ~~2A~~

Consequence: A randomly chosen matrix will have 100% chance of being diagonalizable over $\mathbb{C}$.

Matrix powers are computed in the same way:

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix}$$

$$A = S D S^{-1}$$

$$D = \begin{bmatrix} 1+4i & 0 \\ 0 & 1-4i \end{bmatrix}.$$

$$A^n = S D^n S^{-1}$$

$$1+4i = \sqrt{17}\,(\cos\theta + i\sin\theta)$$

$$\theta = \operatorname{Arctan}(4/1) \approx 75.9^\circ.$$

$$(1+4i)^n = (\sqrt{17})^n\,(\cos(n\theta) + i\sin(n\theta))$$

$$76^\circ \cdot 45 = 180^\circ + 9 \cdot 360^\circ.$$

$$A^{45} = \left(\sqrt{17}\right)^{45} \begin{pmatrix} -0.999595\ldots & -0.0142309 \\ 0.056923 & -0.999595 \end{pmatrix}$$

<u>almost</u>

$$\left(\sqrt{17}\right)^{45} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

rotation by 180°.