Lecture 26

Matrix Powers and Dynamical Systems.

Diagonalization makes computing matrix powers easy.

If $A = SDS^{-1}$ $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

Then:

$$A^k = \left(SDS^{-1}\right)^k$$

$$= (SDS^{-1})(SDS^{-1})\dots(SDS^{-1})$$

$$= SD\underbrace{\cancel{(S^{-1}S)}}^{I}D\underbrace{\cancel{(S^{-1}S)}}^{I}\dots\underbrace{\cancel{(S^{-1}S)}}^{I}DS^{-1}$$

$$= SDD\dots DS^{-1}$$

$$= SD^kS^{-1} = S\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}S^{-1}.$$

Alternatively:

If $v \in \mathbb{R}^n$, compute $A^k v$.

If $v_1, \ldots, v_n$ eigenbasis for A, we can write v in $(v_1, \ldots, v_n)$-coordinates.

$$v = c_1 v_1 + \ldots + c_n v_n.$$

Then

$$\begin{aligned} A^k v &= A^k (c_1 v_1 + \ldots + c_n v_n) \\ &= c_1 A^k v_1 + c_2 A^k v_2 + \ldots + c_n A^k v_n \\ &= c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \ldots + c_n \lambda_n^k v_n \end{aligned}$$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1} v \qquad S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$

Compute $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for all k.

Solution: Diagonalize A (if we can).

① Find eigenvalues

(1A) Compute char poly.

$$\mathrm{Det}(A - tI) = \begin{vmatrix} 2-t & 3 \\ 4 & 3-t \end{vmatrix}$$

$$= (2-t)(3-t) - 12$$

$$= -6 - 5t + t^2.$$

(1B) Find roots.

$$\frac{5 \pm \sqrt{5^2 + 4 \cdot 6}}{2}$$

$$\lambda = -1 \ , \ \lambda = 6.$$

(2) Find eigenbasis

For each eigenvalue λ
Compute λ- eigenspace

$$E_\lambda = \ker(A - \lambda I).$$

Choose basis + concatenate.

$\lambda = -1$

$$A - \lambda I = \begin{bmatrix} 2+1 & 3 \\ 4 & 3+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$$

$$\mathrm{Rref}(A - \lambda I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$E_{-1} = \mathrm{Ker}(\mathrm{Rref}(A - \lambda I))$$

$$= \mathrm{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right).$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$\lambda = 6$

$$A - \lambda I = \begin{bmatrix} 2-6 & 3 \\ 4 & 3-6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}.$$

$$\text{Rref}(A - \lambda I) = \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}.$$

$$E_6 = \text{Ker}(A - \lambda I) = \text{Span}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$$

$$v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

v_1, v_2 is a basis for $\mathbb{R}^2$.

③ Make D and S.

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}.$$

$$S = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

So $A = SDS^{-1}$.

4 Compute S^{-1}.

$$S^{-1} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}.$$

$$\underset{A}{\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}} = \frac{1}{7}\underset{S}{\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}}\underset{D}{\begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}}\underset{S^{-1}}{\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}}$$

(5) Remember the problem.

$$A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (SDS^{-1})^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= S D^k S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

<u>or</u>

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 v_1 + c_2 v_2$$

where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

$$= S^{-1} v$$

$$S^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{7}\begin{bmatrix}4 & -3\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}.$$

$$= \begin{bmatrix}4/7\\1/7\end{bmatrix} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$$

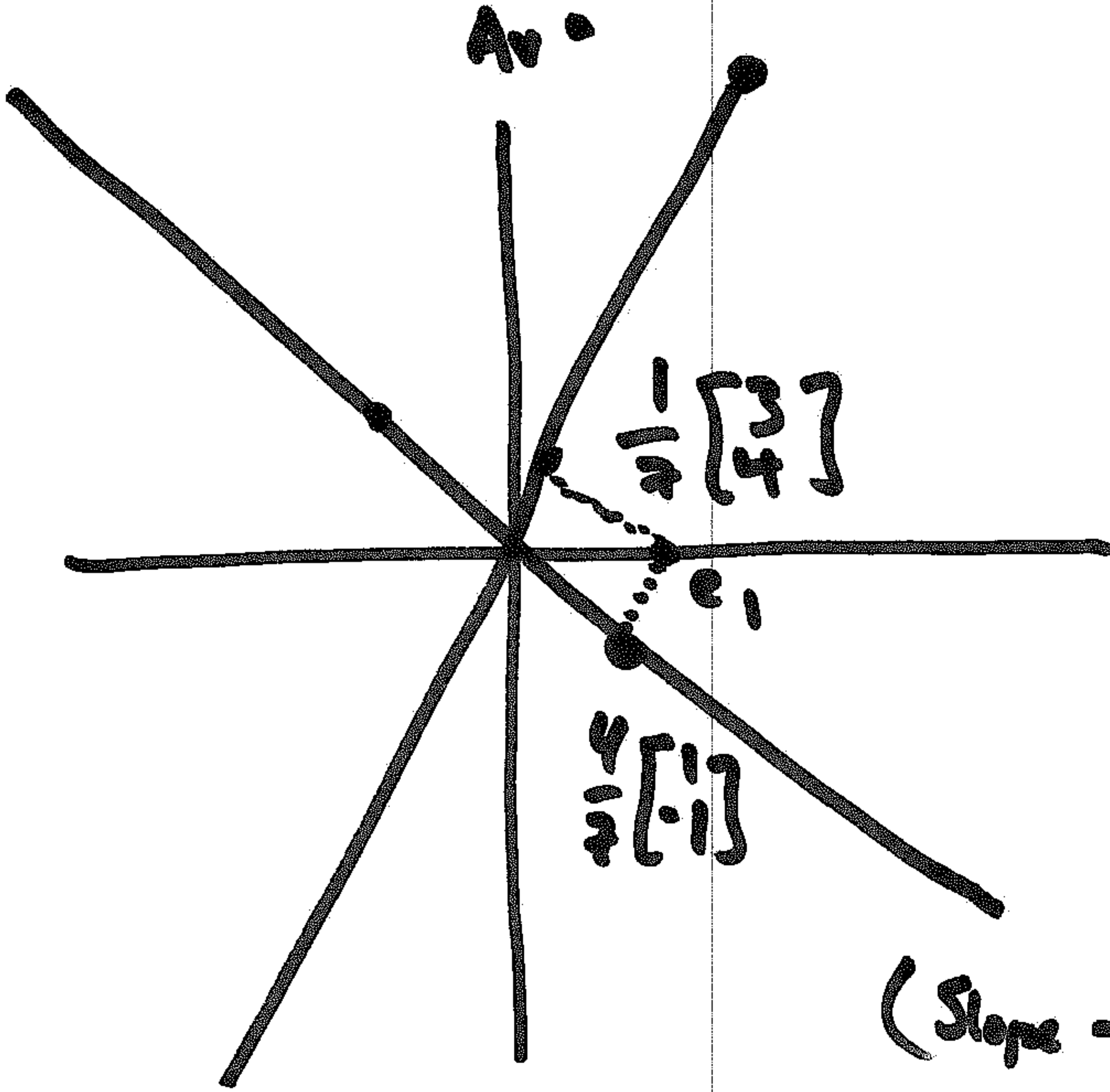
$$\begin{bmatrix}1\\0\end{bmatrix} = \frac{4}{7}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{7}\begin{bmatrix}3\\4\end{bmatrix}.$$

$$A^K\begin{bmatrix}1\\0\end{bmatrix} = \frac{4}{7}A^K\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{7}A^K\begin{bmatrix}3\\4\end{bmatrix}$$

$$= \frac{4}{7}(-1)^K\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{7}6^K\begin{bmatrix}3\\4\end{bmatrix}.$$

END. $\square$

How does A act on $\begin{bmatrix}1\\0\end{bmatrix}$?



(Slope $4/3$)

6 - eigenspace

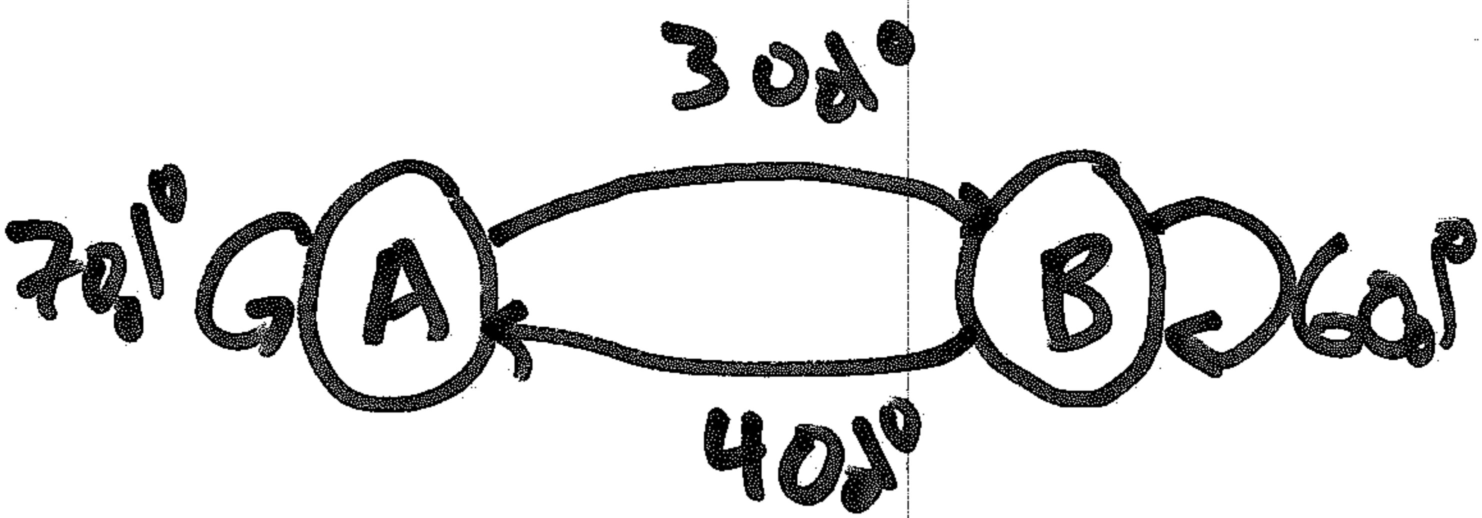
$Av = 6v$

-1 eigenspace

$Av = -v$

Application: Markov Chains.

Imagine particles distributed between two states A and B. From one time to the next, a particle changes state according to the rule:



Find how distribution changes over time.

We may represent "how states change" by a transition matrix:

$$\begin{array}{cl} & \begin{array}{cc} A & \;\; B \end{array} \qquad \left(\begin{array}{c}\text{state}\\ \text{now}\end{array}\right) \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} = T \end{array}$$

$\left(\begin{array}{l}\text{state}\\ \text{1 time}\\ \text{from now}\end{array}\right)$

An initial distribution is given by a vector $v(0)$.

e.g. $v(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 100% at A.

"The system evolves according to T"

$$v(0) \xrightarrow{T} v(1) \xrightarrow{T} v(2) \xrightarrow{T} v(3)$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 0.61 \\ 0.39 \end{bmatrix} \xrightarrow{T}$$

$$\begin{bmatrix} 0.583 \\ 0.417 \end{bmatrix}$$

i.e. $$v(t) = T(v(t-1)) = T^t v(0).$$

Problem: Find formula for distribution

$$v(t) = T^t v(0)$$

at time t.

Compute $T^t v(0)$.

① Char poly of T

$$t^2 - 1.3t + 0.3$$

② Find roots (eigenvalues)

$$1, \ 0.3$$

③ Eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ e.v. } \lambda = 0.3$$

$$\frac{1}{7}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \text{ e.v. } 1$$

(4) write $v(0)$ in eigenvalues coordinates:

$$\begin{bmatrix}1\\0\end{bmatrix} = v(0) = \frac{1}{7}\begin{bmatrix}4\\3\end{bmatrix} + \frac{3}{7}\begin{bmatrix}1\\-1\end{bmatrix}.$$

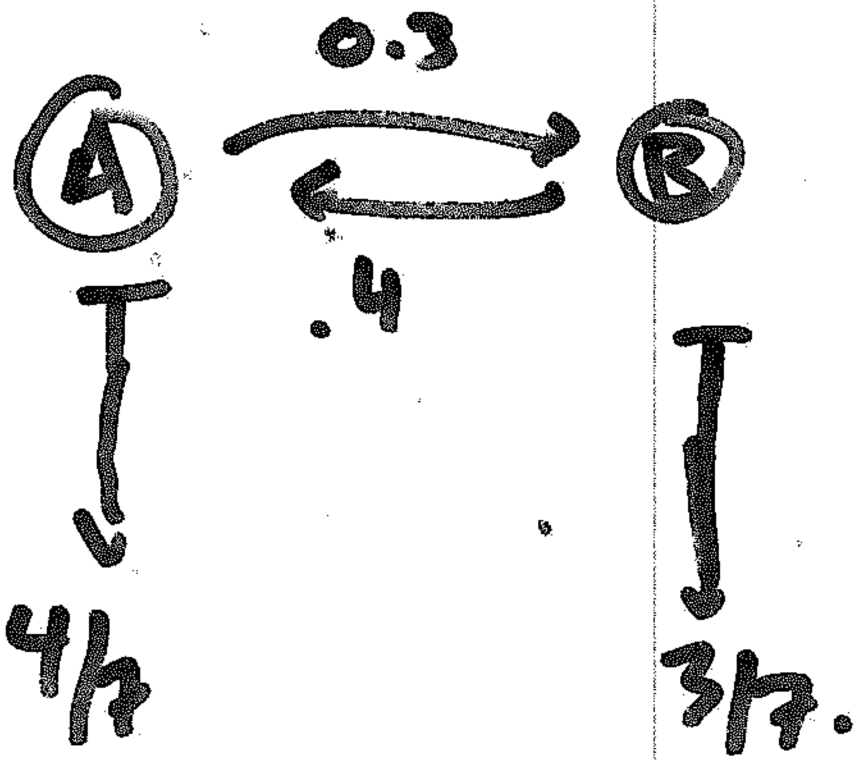
(5)

$$\begin{aligned} v(t) &= T^t v(0) \\ &= \frac{1}{7} T^t \begin{bmatrix}4\\3\end{bmatrix} + \frac{3}{7} T^t \begin{bmatrix}1\\-1\end{bmatrix}. \\ &= \frac{1}{7}(1)^t \begin{bmatrix}4\\3\end{bmatrix} + \frac{3}{7}(0.3)^t \begin{bmatrix}1\\-1\end{bmatrix}. \\ &= \frac{1}{7}\begin{bmatrix}4\\3\end{bmatrix} + \frac{3}{7}(0.3)^t \begin{bmatrix}1\\-1\end{bmatrix}. \end{aligned}$$

As $t \to \infty$

$$\lim_{t \to \infty} v(t) = \frac{1}{7}\begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

This is "the equilibrium" distribution



Def: A transition matrix is an $n \times n$ matrix such that

1. The sum of every entry along a column is 1.
2. All entries $0 \le a_{ij} \le 1$.

if $a_{ij} > 0$ we say A is positive if A^n is positive for some n we say A is regular.

(Positive $\Rightarrow$ regular).

Thm: If A is a regular transition matrix

① 1 is an eigenvalue of A and all eigenvalues λ $|\lambda| \leq 1$.

② There is a unique eigenvector with eigenvalue 1 such that sum of all entries is 1.

x equilibrium.

(3) If $v(0)$ is any vector

such that all entries

sum to 1.

$$\lim_{t \to \infty} A^t(v(0)) = x_{equal}.$$