Lecture 25

Finding the eigenvectors of a matrix

Problem: Diagonalize

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Want

$$A = SDS^{-1}$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \quad S = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

i.e. $AS = SD$.

$$AS = A \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ Av_1 & Av_2 & Av_3 \\ | & | & | \end{bmatrix}$$

$$SD = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \lambda_3 v_3 \\ | & | & | \end{bmatrix}.$$

if $AS = SD$ then

$Av_i = \lambda_i v_i$ in i^{th} column

i.e.

v_1, v_2, v_3 are eigenvectors

w/ eigenvalues $\lambda_1, \lambda_2, \lambda_3$, resp.

Step 1 : Find eigenvalues of A.

(1) Compute char poly of A.

$$P_A(t) = \operatorname{Det}(A - tI)$$

$$= \begin{vmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{vmatrix}$$

$$= -t \begin{vmatrix} -t & 1 \\ 1 & -t \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -t \end{vmatrix}$$

$$+ 1 \cdot \begin{vmatrix} 1 & -t \\ 1 & 1 \end{vmatrix}$$

$$= -t\left((-t)^2 - 1\right) - 1\left(-t - 1\right)$$
$$+ 1\left(1 - (-t)\right).$$

$$= (-t)(t^2 - 1) + (t+1)$$
$$+ (t+1).$$

2 Find the roots (eigenvalues) of $P_A(t)$.

$$P_A(t) = (t+1)\left((-t)(t-1) + 2\right)$$
$$(2 - t + t^2)$$

$$(2 - t + t^2) = (1+t)(2-t).$$

-1 and 2.

$$p_A(t) = (t+1)^2(2-t).$$

Roots are

$\lambda = -1$ with mult 2.

2 with mult 1.

So if A is diagonalizable, then D should be

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The order of eigenvalues doesn't matter, (except when I write down S λ_i should be eigenvalue of i^{th} column)

Step 2: Find eigenvectors for eigenvalues computed in step 1.

$\lambda = -1$. want v such that $Av = -v$ i.e.

$$Av + v = \vec{0}$$
$$(A + I)v = \vec{0}$$

$$v \in \operatorname{Ker}(A + I).$$

Eigenvectors w/ eigenvalue -1 are elements of $\operatorname{Ker}(A+I)$.

Compute $\ker(A+I)$.

$$A + I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Row reduce $A + I$

$$\operatorname{Rref}(A+I) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$x + y + z = 0$$

$\ker(A+I)$

$$= \left\{ \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} \right\}$$

$$= \mathrm{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right).$$

2 dimensional space of eigenvectors w/ eigenvalue -1.

Choose a basis for this space v_1, v_2

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Compute all eigenvectors with eigenvalue $2 = \lambda$.

$$Av = 2v \iff v \in \operatorname{Ker}(A - 2I)$$

$$A-2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\mathrm{Rref}(A-2I) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x - z = 0$$
$$y - z = 0$$

$$\mathrm{Ker}(A-2I) = \left(\begin{bmatrix} z \\ z \\ z \end{bmatrix} \right) = \mathrm{Span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

everything in $\ker(A - 2I)$
eigenvector w/ eigenvalue 2 example:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Choose a basis

$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for this space.

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

eigenvects with eigenvalues

$-1, -1, 2$. These are a basis for $\mathbb{R}^3$.

$$S = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$ is invertible.

$$A = SDS^{-1}$$

$$S = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \quad \square$$

Let's give names to thing that arose during this process:

Def: Let A be an $n\times n$ matrix, the λ-eigenspace of A is

$$E_\lambda = \{v \in \mathbb{R}^n \mid Av = \lambda v\}$$
$$= \mathrm{Ker}(A - \lambda I).$$

~~Step 2~~ In step 2, we computed the eigenspaces E_λ for each eigenvalue λ of A. Found a basis for each.

To write down the matrix S, we need at least algmult(λ) many linearly independent eigenvects with eigenvalue λ.

Def: The geometric multiplicity of an eigenvalue λ is the dimension of the λ-eigen-space.

$$\text{geomult}(\lambda) = \dim(\ker(A-\lambda I))$$

To diagonalize A, we need that $\text{geomult}(\lambda) = \text{alg}(\lambda)$ for each eigenvalue λ.

Example (Alg v. Geometric mult)

A

eigenvalues

	alg mult	geo.
-1	2	2
2	1	1

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad p_A(t) = (1-t)^2$$

eigenvalues	alg mult	geomult
1	2	1

$$\operatorname{Ker}(A - I) = \operatorname{Ker}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

General Process to diagonalize A

① Compute char poly A and find roots.

(These are eigenvalues for A).

! For A to be diagonalizable all roots must be real numbers.

This gives you $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

② Find a basis of eigenvects.

For each eigenvalue λ compute a basis for the λ- eigenspace

$v_1, \dots, v_k$

! If $k =$ geomult(λ) is not equal to algmult(λ), A is <u>not</u> diagonalizable.

③ Concatenate these bases for eigenspaces

$v_1, \dots, v_k, v_1', \dots, v_{k'}', \dots$

Thm: if geomult(λ) = alg(λ) for all λ, then there are n vectors in this list and this list is a basis for $\mathbb{R}^n$

$$S = \begin{bmatrix} | & | & & | & | & \\ v_1 & v_2 & \cdots & v_k & v_1' & \cdots \\ | & | & & | & | & \end{bmatrix}$$

$$A = SDS^{-1}.$$