Lecture 24

Finding the eigenvalues of a matrix.

Def: An eigenvalue of a square matrix A is a number $\lambda \in \mathbb{R}$ such that

$$Av = \lambda v$$

for some <u>non-zero</u> vector $v \in \mathbb{R}^n$.

Example: $A = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

1 and 3 are eigenvalues of A.

Amazing fact we can find eigenvalues w/o knowing eigenvectors!

Thm: Let A be an $n \times n$ matrix and consider the function $P_A : \mathbb{R} \longrightarrow \mathbb{R}$

$$P_A(t) = \operatorname{Det}(A - tI).$$

Then $P_A(\lambda) = 0$ if and only if λ is an eigenvalue of A.

Example $A = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$.

$$A - tI = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} - t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 5-t & -2 \\ 4 & -1-t \end{pmatrix}.$$

$$p_A(t) = \begin{vmatrix} 5-t & -2 \\ 4 & -1-t \end{vmatrix}$$

$$= (5-t)(-1-t) - (-2)\cdot 4$$

$$= -5 - 4t + t^2 + 8$$

$$= t^2 - 4t + 3.$$

$$= (t-1)(t-3).$$

Ex 1: $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

Compute $p_A(t)$. $A - tI$.

(Subtract t from each diagonal entry)

$$\operatorname{Det}(A - tI) = \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 2-t & 1 \\ 0 & 0 & 3-t \end{vmatrix}$$

$$= (1-t)(2-t)(3-t)$$

Upper triangular $\Rightarrow$ det is product of diagonal entries.

$p_A(t) = (1-t)(2-t)(3-t)$

Roots of $p_A(t)$

$1, 2, 3.$

Eigenvalues of A are

$1, 2, 3.$

Thm: The eigenvalues of an upper triangular matrix

$$A = \begin{pmatrix} a_{11} & * & * \\ & \ddots & * \\ 0 & & a_{nn} \end{pmatrix}$$

are the diagonal entries $a_{11}, \dots, a_{nn}$. The char. poly. is

$$p_A(t) = \prod_{i=1}^{n} (a_{ii} - t).$$

Ex 3: $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

$$P_A(t) = \operatorname{Det}(A - tI)$$

$$= \begin{vmatrix} 1-t & 0 & 1 \\ 0 & 1-t & 0 \\ -1 & 0 & 1-t \end{vmatrix}.$$

Compute via laplace expansion.

(Along First row)

$$\mathrm{Det}(B) = b_{11}\,\mathrm{Det}(B_{11}) - b_{12}\,\mathrm{Det}(B_{12}) + b_{13}\,\mathrm{Det}(B_{13}).$$

For our case

$$(1-t)\begin{vmatrix} 1-t & 0 \\ 0 & 1-t \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ -1 & 1-t \end{vmatrix} + 1\begin{vmatrix} 0 & 1-t \\ -1 & 0 \end{vmatrix}.$$

$p_A(t) =$

$(1-t)^3 - 0 + 1(1-t)$

$= (1-t)^3 + (1-t).$

$= (1-t)\left[(1-t)^2 + 1\right].$

1 is a root | $1 \pm \sqrt{-1}$ root of $(1-t)^2 + 1.$

1 is a root of $p_A(t) \implies 1$ is an eigenvalue of A. (only one).

What type of char. polys. do we expect when A is diagonalizable?

obs:

$$A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

$$P_A(t) = \prod_{i=1}^{n} (a_i - t).$$

So $P_A(t)$ has n real roots.

Thm: If $A = SBS^{-1}$ then $p_A(t) = p_B(t)$

(Similar matrices have the <u>same</u> char poly).

Cons. If A is diagonalizable then $p_A(t)$ must have n real roots.

(Example 3) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

is <u>NOT</u> diagonalizable (only has 1 real root).

Why is thm true?

$$P_A(t) = \operatorname{Det}(A - tI)$$
$$= \operatorname{Det}(SBS^{-1} - tI)$$
$$= \operatorname{Det}(SBS^{-1} - tSS^{-1})$$
$$= \operatorname{Det}(SBS^{-1} - S(tI)S^{-1})$$
$$= \operatorname{Det}(S(B - tI)S^{-1})$$

$\operatorname{Det}(CD) = \operatorname{Det}(C) \cdot \operatorname{Det}(D)$

$$= \operatorname{Det}(S)\operatorname{Det}(B - tI)\operatorname{Det}(S^{-1})$$
$$= \operatorname{Det}(S)\ P_B(t) \cdot \operatorname{Det}(S)^{-1}$$
$$= \underbrace{\operatorname{Det}(S)\operatorname{Det}(S)^{-1}}_{1}\ P_B(t)$$
$$= P_B(t).$$

Summary:

- Roots of $p_A(t)$ are eigenvalues of A.
- For A to be diagonalizable it is necessary (but not sufficient) for $p_A(t)$ to have all real roots

$$p_A(t) = \prod_{i=1}^{n} (\lambda_i - t)$$

λ_i real.

- If A is diagonalizable and $p_A(t) = \prod_{i=1}^{n} (\lambda_i - t)$

then

$$A \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Ex 4: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$p_A(t) = (1-t)^2$$

I want to say $p_A(t)$ has <u>two</u> real roots. Not true

1 is the only root. But it's a root "twice."

Def: The algebraic multiplicity of an eigenvalue λ is the number of times $(t-\lambda)$ divides $p_A(t)$, i.e. if

$$p_A(t) = (t-\lambda)^K g(t)$$

$g(\lambda) \neq 0$, then K is the algebraic multiplicity of λ.

Thm: For a matrix to be diagonalizable it is necessary that $p_A(t)$ has $\underline{n}$ real roots counting multiplicity.