Lecture 23

Diagonalization.

Def: A diagonal matrix is a square matrix whose only non-zero entries occur along the (main) diagonal

e.g. $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

all zeros

Diagonal matrices scale coordinate axes by diagonal entries

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot x \\ 1 \cdot y \\ 2 \cdot z \end{pmatrix}$$

Linear algebra with diagonal matrices is easy!

A	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\operatorname{Det}(A)$	$0 \cdot 1 \cdot 2 = 0$	$3 \cdot 1 \cdot 1 = 3.$
$\operatorname{Ker}(A)$	$\operatorname{Span}(e_1)$	$\{0\}.$
$\operatorname{Im}(A)$	$\operatorname{Span}(e_2, e_3)$	$\mathbb{R}^3.$
A^{-1}	Does Not exist	$\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \begin{pmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{pmatrix}$

A^n	$\begin{pmatrix}0^n & & \\ & 1^n & \\ & & 2^n\end{pmatrix}$	$\begin{pmatrix}3^n & & \\ & 1^n & \\ & & 1^n\end{pmatrix}$
	$\begin{pmatrix}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n\end{pmatrix}$	$\begin{pmatrix}3^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}.$

- To invert, invert entries along diagonal
- To take powers, take powers of diagonal entries.

Diagonalization is the process of finding a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ so that $[A]_{v_1, \ldots, v_n}$ is a diagonal matrix.

(warning: such a basis doesn't always exist)

Def: A linear transformation $A: \mathbb{R}^n \to \mathbb{R}^n$ is said to be diagonalizable, if there exists a basis $\beta = (v_1, .., v_n)$ such that

$$[A]_\beta \text{ is}$$

a diagonal matrix.

Equivalently: if there is an invertible matrix

$$S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

such that

$$S^{-1} A S = D$$

a diagonal matrix.

Examples

(1) Choose S, D calculate $A = SDS^{-1}$.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

$$S = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

$$S^{-1} = \frac{1}{\det(S)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

$$A = \overset{S}{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}} \overset{D}{\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}} \overset{S^{-1}}{\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

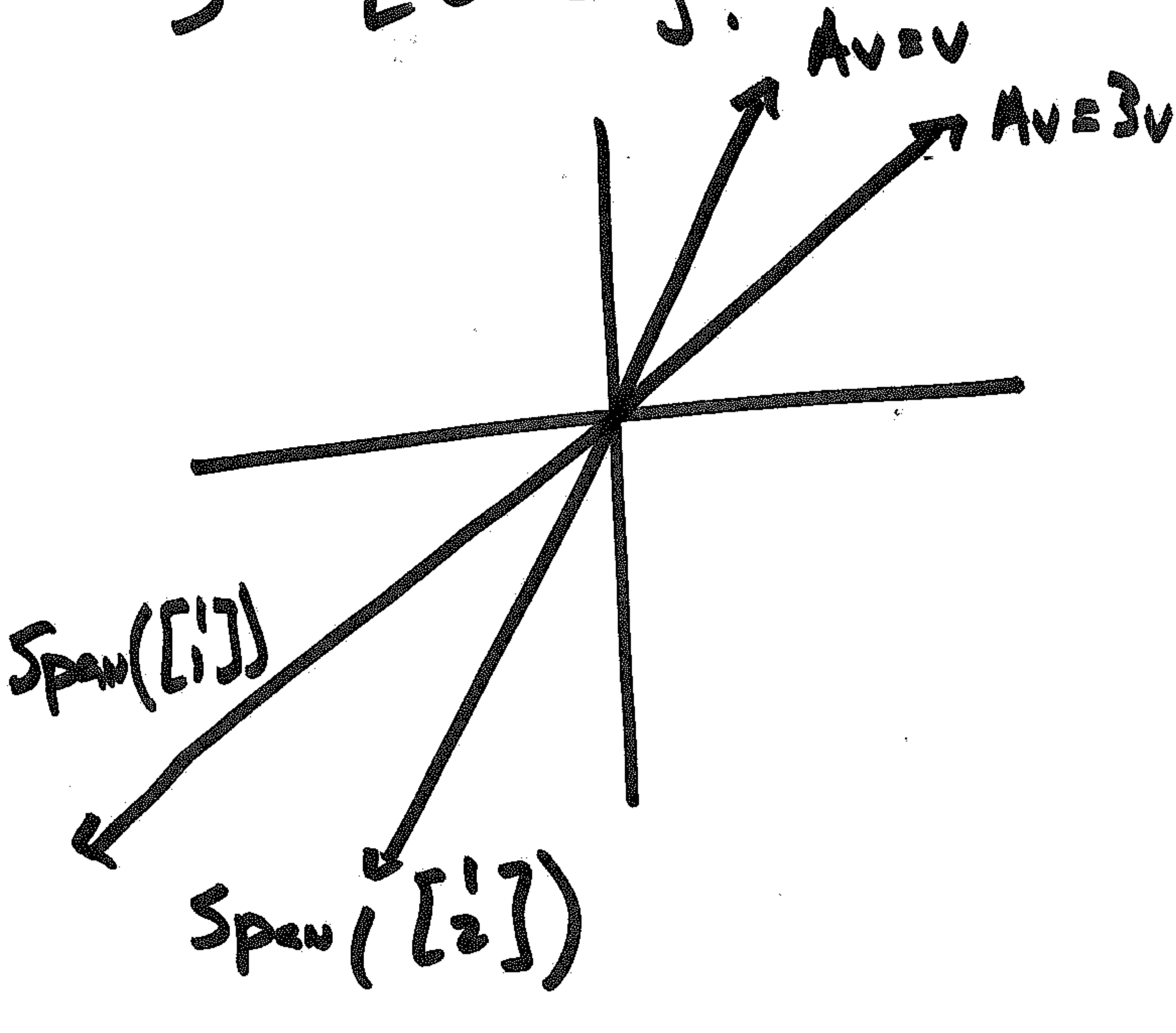
$$= \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}.$$

$A = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$ is

diagonalizable.

A in $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

coordinates is given by $\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$.



② Orthogonal projections are diagonalizable.

V subspace

$v_1, \dots, v_k$ basis V.

and

$v_{k+1}, \dots, v_n$ basis $V^\perp$.

Then $v_1, \dots, v_n$ is a basis for $\mathbb{R}^n$.

Then

$$\left[\mathrm{Proj}_{V'} \right]_{v_1,\dots,v_n} = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{k\text{-times}} & 0 \\ 0 & \underbrace{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}}_{n-k \text{ times}} \end{pmatrix}$$

When a matrix is diagonalizable, its determinant, powers, inverse, kernel, and image can be calculated from its diagonalization (and S)

Thm: If $A = SBS^{-1}$, then

① $\mathrm{Det}(A) = \mathrm{Det}(B)$.

② $A^n = SB^nS^{-1}$

③ $A^{-1} = SB^{-1}S^{-1}$

(if A or B is invertible).

Example:

$$A = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}.$$

$$A = S D S^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

$$\det(A) = 5(-1) - 4(-2)$$

1 $$= 3.$$

$$\det(D) = 1 \cdot 3 = 3.$$

Powers:

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot 3^n - 1 & 1 - 3^n \\ 2 \cdot 3^n - 2 & 2 - 3^n \end{pmatrix}.$$

Linear algebra with
A is easy if
we know how to
diagonalize it.

To diagonalize a matrix
A we need to find
a basis which transforms
in specific way.

A diagonal matrix has the property that

$$Ae_i = \lambda_i e_i$$

where

$$\left[A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \right]$$

for some scalar λ_i.

To diagonalize a matrix, we need to find a basis $v_1, \ldots, v_n$ such that

$$Av_i = \lambda_i v_i$$

for some scalar λ_i

(diagonalizable matrices preserve and scale coordinate axes in β-coordinates)

Def: Let A be a square matrix. A non-zero vector $v \in \mathbb{R}^n$ is called an <u>eigenvector</u> of A if

$$Av = \lambda v$$

for scalar λ. The scalar λ is called the <u>eigenvalue</u> associated to the eigenvector v.

Ex $A = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ $S = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

$A = SDS^{-1}$.

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

eigenvectors of A

with eigenvalues 1 and 3,

respectively.

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5-4 \\ 4-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= 1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$= 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thm: An $n \times n$ matrix A is diagonalizable if and only if there is a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ consisting of eigenvectors for A. Such a basis (if it exists) is called an eigenbasis.