Lecture 22: Determinants II

- Laplace Expansion
- Cramer's Rule

Notation: A matrix A with vertical bars means $\det(A)$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= ad - bc.$$

$$\begin{vmatrix} \ddots & & \\ & a_{ij} & \\ & & \ddots \end{vmatrix} = \det \begin{bmatrix} \ddots & & \\ & a_{ij} & \\ & & \ddots \end{bmatrix}.$$

Problem:

Compute $\operatorname{Det}\begin{pmatrix} x & 1 & 2 \\ y & 3 & 4 \\ z & 5 & 6 \end{pmatrix}$.

Solution: we could do this by row reduction.

Here's another way:

$$\begin{vmatrix} x & 1 & 2 \\ y & 3 & 4 \\ z & 5 & 6 \end{vmatrix} = \begin{vmatrix} x & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 \\ y & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ z & 5 & 6 \end{vmatrix}$$

by linearity $$= x\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} + y\begin{vmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} + z\begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix}$$

Now we have to compute the determinants of matrices all whose entries are numbers.

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & \boxed{3} & 4 \\ 0 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & & \\ & r_1 & \\ & & r_2 \end{vmatrix}$$

$$r_1 r_2 = \det\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

row reduce this matrix until I get a diagonal matrix.

$$= \det\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

$$= -2.$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 5 & 6 \end{vmatrix}$$

$$= - \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$$

$$= -(6-10)$$

$$= 4.$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 1 & 5 & 6 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 5 & 6 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{vmatrix}$$

Switch rows until

$$= (-1)^2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= (4-6)$$

$$= -2.$$

Putting this all together

$$\begin{vmatrix} x & 1 & 2 \\ y & 3 & 4 \\ z & 5 & 6 \end{vmatrix} = x \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} + y \left| \;\; '' \;\; \right| + z \left| \;\; \right.$$

$$= x \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + y(-1) \left| \begin{matrix} 1 & 2 \\ 5 & 6 \end{matrix} \right]$$

$$+ (-1)^2 z \left| \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} \right]$$

$$\boxed{= -2x + 4y + -2z.}$$

Def: Let A be an $n \times n$ matrix, the ij-th <u>Minor</u> of A is the $(n-1) \times (n-1)$ matrix A_{ij} obtained by deleting the i^{th}-row and j-th column from A.

$$\begin{bmatrix} \cancel{\ast} & \cancel{1} & \cancel{2} \\ \cancel{\ast} & 3 & 4 \\ \cancel{\ast} & 5 & 6 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} \cancel{\ast} & 1 & 2 \\ \cancel{\ast} & \cancel{3} & \cancel{4} \\ \cancel{\ast} & 5 & 6 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}.$$

IN our example:

$$\mathrm{Det}\begin{pmatrix} x & 1 & 2 \\ y & 3 & 4 \\ z & 5 & 6 \end{pmatrix}$$

$$= \det(A_{11})x - \det(A_{21})y + (-1)^2 \det(A_{31})z$$

This works for <u>any</u> 3x3 matrix.

This generalizes to $n \times n$ matrices:

Let A be an $n \times n$ matrix and i be between $1 \leq i \leq n$ (i th row) then the laplace expansion along the ith row is:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(A_{ij})$$

$\det(A_{ij})$

$$A = \begin{bmatrix} & a_{ij} & \\ & & \end{bmatrix}$$

the laplace expansion along the j^{th} column is

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

"Example"

$$\begin{array}{cc} i^{th} & \begin{pmatrix} \cancel{a_{i1}} & \cancel{a_{i2}} & \cdots \\ & & \\ & & \end{pmatrix} \end{array}$$

(1) Find A_{ij} th minor for every entry of i^{th} row.

(2) Compute determinant.

(3) Scale by a_{ij} and sum according to the signs

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & & & & \end{pmatrix}$$

Remarks:

- Laplace expansion is a recursive formula for det(A).

Neg
- Computing the det via Laplace expansion is slow.

Pos
- Laplace expansion doesn't require any division.

Let A be an $n \times n$ matrix

consider

$$\mathrm{Adj}(A) = \begin{bmatrix} (-1)^{i+j}\det(A_{ji}) \\ \\ \end{bmatrix}$$

ij-th entry is $(-1)^{i+j}\det(A_{ji})$.

(note ij and ji are switched in entry v. A_{ji}).

Cramer's Rule:

If $\det(A) \neq 0$, then
$$\frac{1}{\det(A)} \mathrm{Adj}(A) = A^{-1}.$$

i.e. formula for inverse in terms of all <u>minors</u>.

Example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$A_{11} = [d]. \qquad A_{12} = [c]$$

$$A_{21} = [b]. \qquad A_{22} = [a].$$

$$\frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$

Why is true?

This what we want.

$$\frac{1}{\det(A)}\begin{bmatrix} & & \\ & (-1)^{i+j}\det(A_{ji}) & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ 0 & 0 & & \ddots & \\ & & 0 & & 1 \end{bmatrix}$$

First row times First column:

First row of $\frac{1}{\det(A)} \mathrm{Adj}(A)$ is

$$\frac{1}{\det(A)}\begin{bmatrix} \det(A_{11}) & (-1)\det(A_{12}) & \dots \end{bmatrix}$$

First column A is:

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ \vdots \\ a_{1n} \end{bmatrix}.$$

What is the product:

$$\frac{1}{\det(A)} \sum \det(A_{1i})\, a_{1i} (-1)^{i+1}$$

By laplace expansion, (along first column)
this is $\dfrac{\det(A)}{\det(A)} = 1.$

Similarly multiply ith column by ith row is 1 by laplace expansion along ith column.

This shows:

$$\frac{\operatorname{Adj}(A)}{\det(A)} \cdot A = \begin{bmatrix} \boxed{1} & & & ? \\ & \boxed{1} & & \\ ? & & \boxed{1} & \\ & & & \ddots & \\ & & & & \boxed{1} \end{bmatrix}$$

~~Notat~~

Let's think about $A = \begin{bmatrix} x & 1 & 2 \\ y & 3 & 4 \\ z & 5 & 6 \end{bmatrix}$.

We know:

$$\begin{bmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \det(A) = 0,$$

if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right)$

because A has a repeated column.

IN particular first row times 2^{nd} + 3^{rd}

column is 0.

(Same thing happens in

general)