Lecture 21

Computing the determinant.

Def: Let $\mathbb{R}^{n \times n}$ be the set of $n \times n$ matrices.

There is a unique function

$$\mathrm{Det}: \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$$

such that

(1) $|\mathrm{Det}(A)| = V(v_1, \ldots, v_n)$

where $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$.

(2) $\mathrm{Det}(I) = 1$.

(3) For fixed column vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ (don't fix ith column) the function

$$T(\vec{x}) = \mathrm{Det}\begin{bmatrix} | & | & & | & & | & & | \\ v_1 & v_2 & \cdots & v_{i-1} & x & v_{i+1} & \cdots & v_n \\ | & | & & | & & | & & | \end{bmatrix}$$

from $\mathbb{R}^n$ to $\mathbb{R}$ is linear

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$$

$$T(K\vec{x}) = K\,T(\vec{x}).$$

Example of prop. (3)

For $A = \begin{bmatrix} 3 & x \\ 7 & y \end{bmatrix}$

$\text{Det}(A) = 3y - 7x$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 3y - 7x.$

This is linear.

In other words,

$$T(x) = [a_1\ a_2 \ldots a_n]x$$

$$= v \cdot x$$

$$v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

For fix $v_1, \ldots, v_n$ (not includ v_i) the determinant

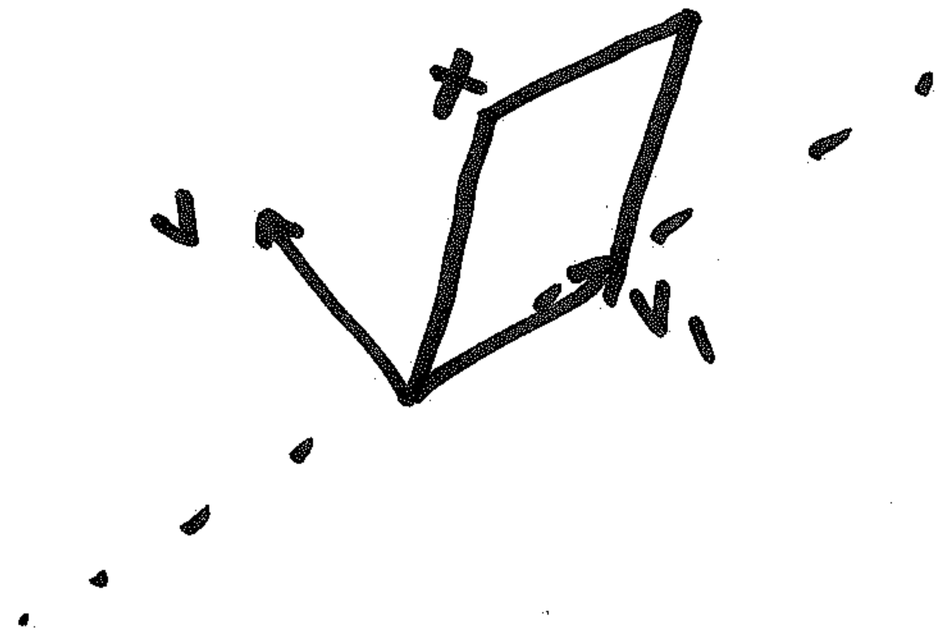
$$\det\left(\left[v_1 \ldots v_{i-1}\ x\ v_{i+1} \ldots v_n\right]\right) = v \cdot x.$$

Example: For

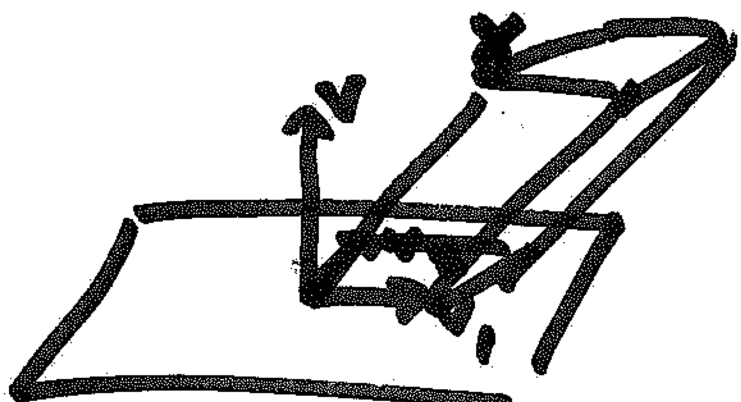
$$A = \begin{bmatrix} 3 & x \\ 7 & y \end{bmatrix}$$

$$v = \begin{bmatrix} -7 \\ 3 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\mathrm{Det}(A) = v \cdot \vec{x}$$



In higher dimensions



$\mathrm{Span}(v_1, \dots, v_n)$
(not including i)

$\vec{v}$ is indicating a direction perpendicular to the plane $\text{Span}(v_1, \ldots, v_n)$ No v_i

which is the "up" direction.

$\not\exists$

If the image of the unit cube under

$$A = [x_1, \ldots, v_{i-1}\ x\ v_{i+1} \ldots v_n]$$

is above the plane the $\det(A) > 0$ and if it's below the plane $\det(A) < 0$.

The determinant is the signed volume of the image of the unit cube under A.

Thm: $\operatorname{Det}(A) \neq 0$

if and only

if A is invertible.

§2. Compute the determinant via reduction.

- Compute via bookkeeping

- Fact: $\operatorname{Det}(A) = \operatorname{Det}(A^T)$

everything we've said about columns is true about rows.

How the determinant changes under row reduction

Rules:

1. Scale a row by K, Scale the determinant by K. (Same is true for a column)

$$\mathrm{Det}\begin{pmatrix} - & Kr & - \end{pmatrix} = K\,\mathrm{Det}\begin{pmatrix} - & r & - \end{pmatrix}$$

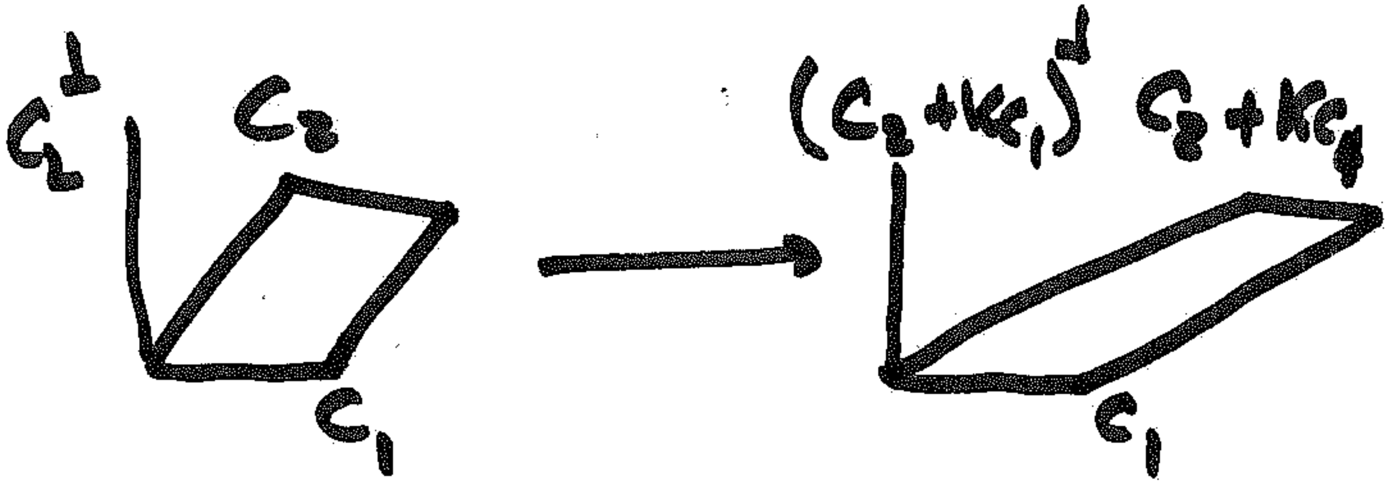
$$\mathrm{Det}\begin{pmatrix} & | & \\ & Kc & \\ & | & \end{pmatrix} = K\,\mathrm{Det}\begin{pmatrix} & | & \\ & c & \\ & | & \end{pmatrix}$$

(2) Add a multiple of a row to another row, determinant stays the same.

$$\operatorname{Det}\begin{pmatrix} - & r_1 & - \\ - & r_2 & - \end{pmatrix} = \operatorname{Det}\begin{pmatrix} - & r_1 + k r_2 & - \\ - & r_2 & - \end{pmatrix}$$

(Same for columns).

$$A = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}$$



Same signed volume

(Same base + same signed height).

(3) Switch two rows, then the sign of determinent flips

$$\mathrm{Det}\begin{bmatrix} - & r_i & - \\ - & r_j & - \end{bmatrix} = -\mathrm{Det}\begin{bmatrix} - & r_j & - \\ - & r_i & - \end{bmatrix}$$

Same with columns.

Idea to compute Det(A)

- row reduce A keeping track of how determinant changes

- Once I obtain Rref(A) then either

$Rref(A) = I$

($det = 1$)

or

$Rref(A) \neq I \Rightarrow det(A) = 0$

Example: Compute Det $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xRightarrow{I \leftrightarrow II} A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

3 $\det(A) = -\det(A_2)$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \xRightarrow{III - I} A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\det(A_2) = \det(A_3)$$

$$\det(A) = -\det(A_3).$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{\mathrm{III} - 2\mathrm{II}} A_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(A_4) = \det(A_3).$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\mathrm{III}/2} A_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$2\det(A_5) = \det(A_4)$$

$$A_5 \xrightarrow{\mathrm{I} - \mathrm{III}} I.$$

$$\det(A_5) = \det(I) = 1.$$

$$\det(A) = -2\det(I)$$
$$= -2.$$

Thm: $\mathrm{Det}(A) = (-1)^s k_1 \ldots k_d$

where s is the # of switches during row reduction !

$k_1, \ldots, k_d$ are constants you divide by.

Thm: Let $A = \begin{bmatrix} x_1 & * & * \\ & \ddots & * \\ 0 & & x_d \end{bmatrix}$

be an upper triangular matrix.

Then $\det(A) = x_1 x_2 \cdots x_d$.

Why? If one of x_i's is 0 then A is not invertible so both $x_1 x_2 \cdots x_d = 0$ and $\det(A) = 0$

Otherwise to row reduce A, divide by $x_1, \ldots, x_d$ in appropriate rows to get 1's along diagonal and then do 2 $\frac{d-1}{2}$ times to kill all stars.

$$\mathrm{Det}(A) = x_1 x_2 \ldots x_d.$$

Thm: $\det(AB) = \det(A)\det(B)$

why?

Row reduce

$$A \xrightarrow{s_1} A_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} I$$

(Consider the case $\det(A) \neq 0$).

Claim:

$$AB \xrightarrow{s_1} A_1B \xrightarrow{s_2} \cdots \xrightarrow{s_n} IB = B$$

$$B \xrightarrow{s_1'} B_1 \to \cdots \to I.$$

(Consider $\det(B) \neq 0$).

So assuming the claim

$$\det(A) = (-1)^{s} k_1 \cdots k_d$$

$$\det(B) = (-1)^{s'} k_1' \cdots k_d'$$

$$\det(AB) = (-1)^{s+s'} k_1 \cdots k_d \, k_1' \cdots k_d'$$

why is the claim true

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} B = \begin{bmatrix} r_1 B \\ \vdots \\ r_n B \end{bmatrix}.$$

(row operations commute with multiply on right).

$$\det(A)\det(A^{-1}) = \det(AA^{-1})$$
$$= \det(I)$$
$$= 1.$$

Thus:

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$