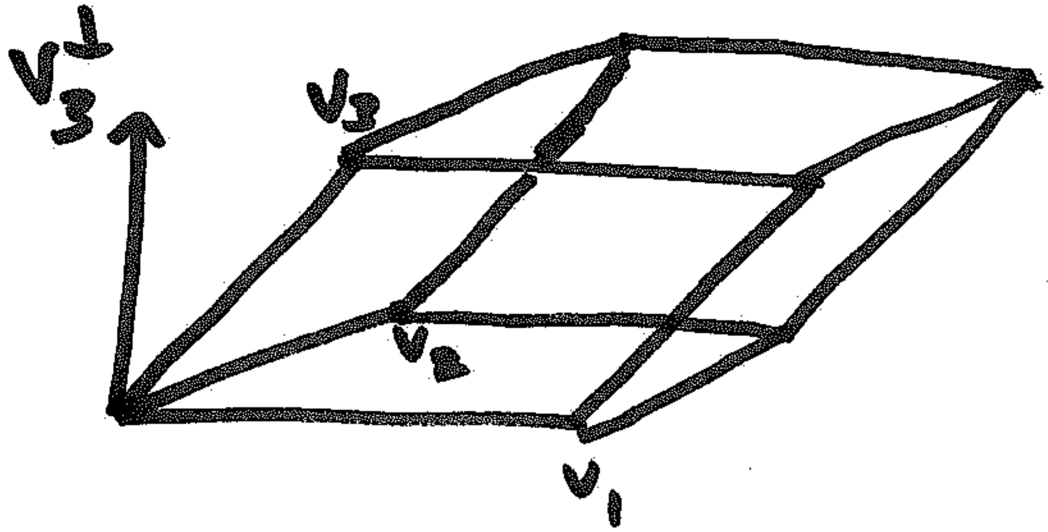
# Lecture 20

## Volumes of Parallelpipeds in $\mathbb{R}^n$.

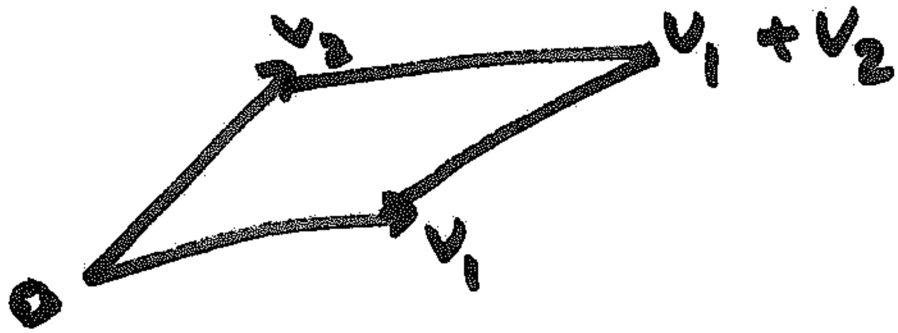



$$\|v_1\| \, \|v_2^\perp\| \, \|v_3^\perp\|$$

# §1. Parallelpipeds.

Parallelpipeds are the higher dimensional generalizations of <u>parallelograms</u>.

A parallelogram:




$$\{ c_1 v_1 + c_2 v_2 \mid 0 \leq c_i \leq 1 \}.$$

Def: Let $v_1, \ldots, v_m \in \mathbb{R}^n$

the <u>m-parallelpiped</u>

defined by $v_1, \ldots, v_m$ is

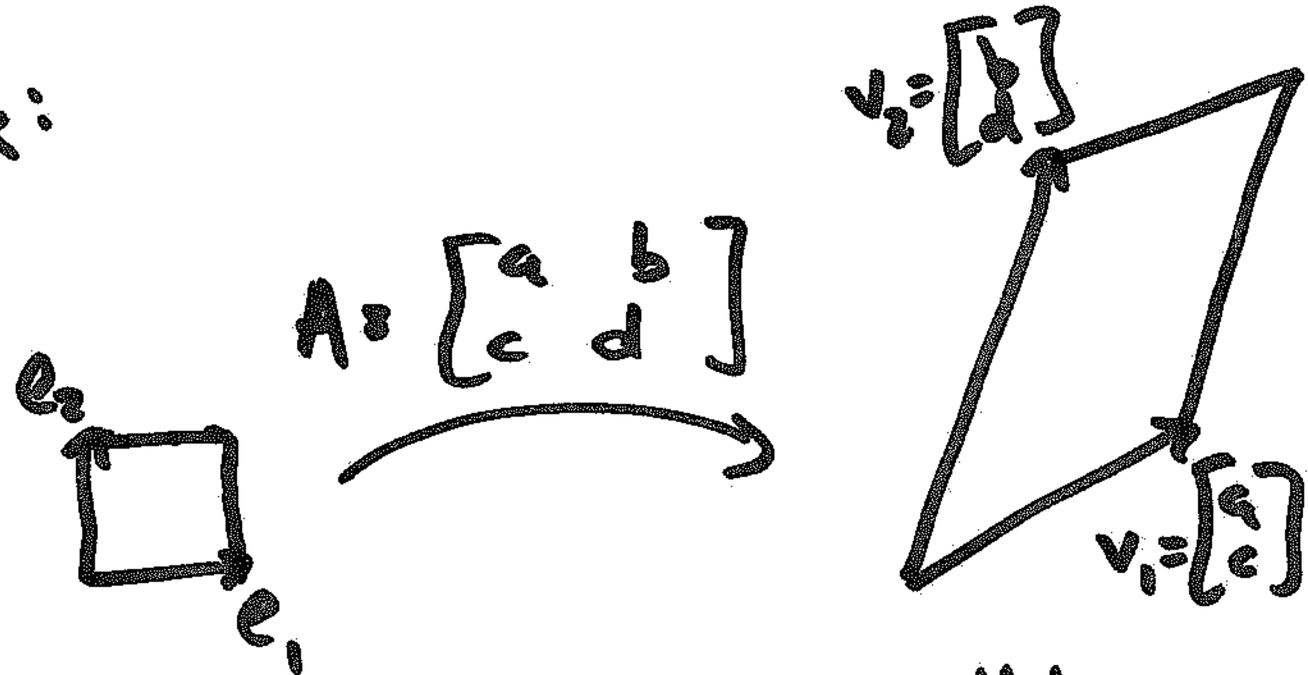
$$\left\{ c_1 v_1 + c_2 v_2 + \ldots + c_m v_m \;\middle|\; 0 \leq c_i \leq 1 \right\}.$$

Example: The $n$-parallelepiped defined by $e_1, \ldots, e_n$ in $\mathbb{R}^n$ is called the <u>unit cube in $\mathbb{R}^n$</u>.

Thm: A linear transformation $A: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ transforms $k$-parallelpipeds into $k$-parallelpipeds.

A matrix transforms the unit cube into the parallel-piped defined by the columns of $A$.

Ex:

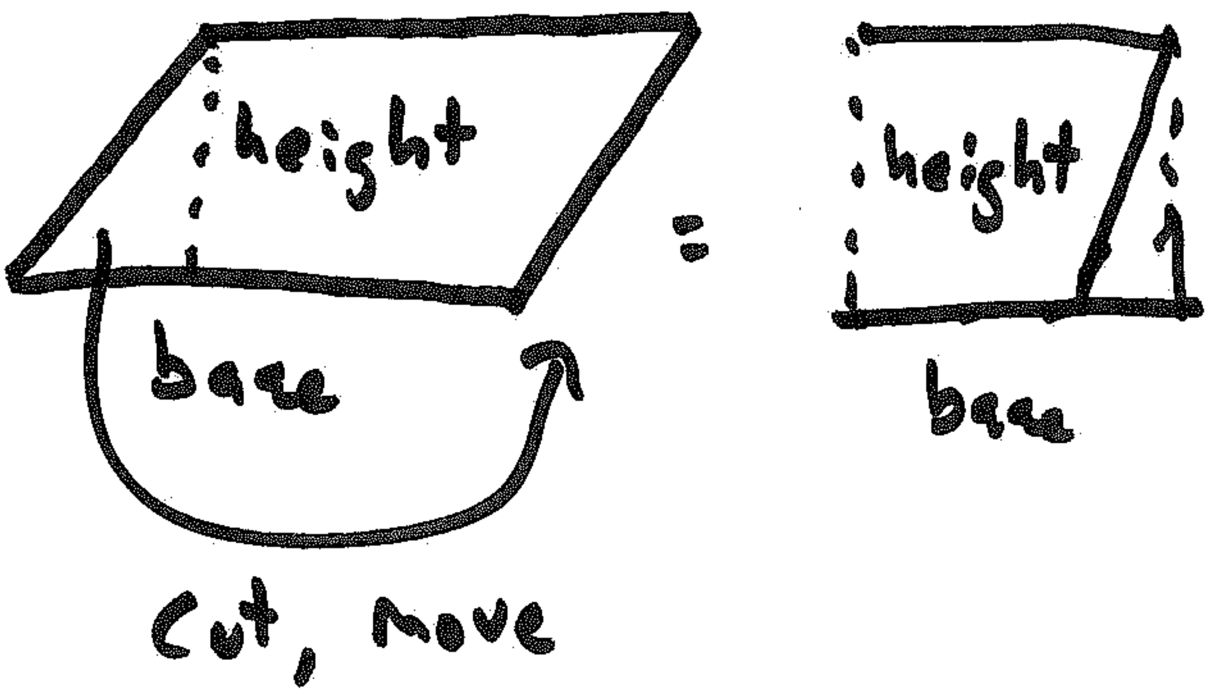



unit square

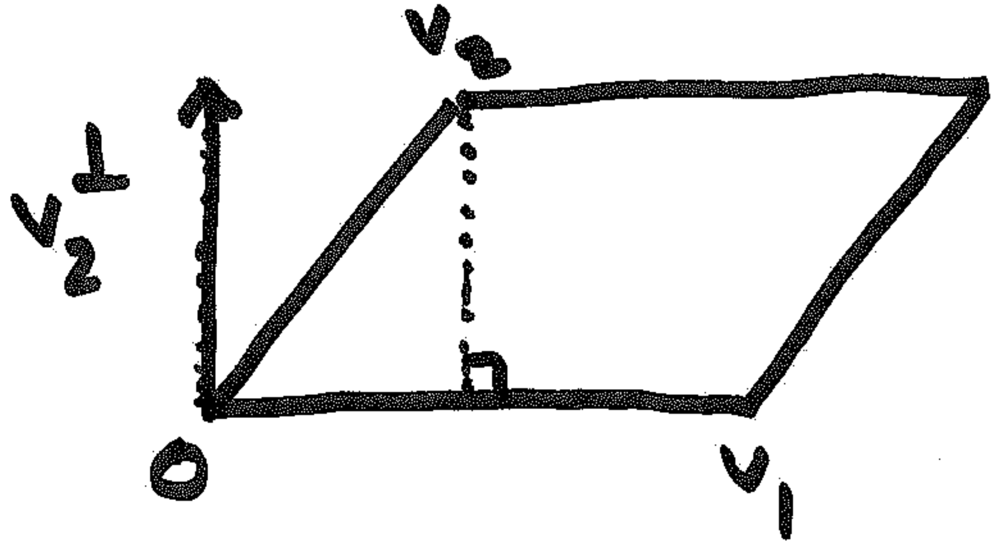
parallelogram defined by the columns of A

Today, I'll talk about the volumes of parallelpipeds.

Parallelogram:




Area = base · height.




base $= \|v_1\|$

height $= \|v_2^{\perp}\|$

Area $= \|v_1\| \, \|v_2^{\perp}\|$.

(2-volume)

Def: The $m$-volume of a $m$-parallelpiped defined by $v_1, \ldots, v_m$ is defined recursively as

$$V(v_1, \ldots, v_m) := V(v_1, \ldots, v_{m-1}) \| v_m^\perp \|$$

where $v_m^\perp \in \mathrm{Span}(v_1, \ldots, v_{m-1})^\perp$.

volume = (base) · height

equivalently

$$V(v_1, \ldots, v_m) = \|v_1\| \, \|v_2^\perp\| \, \|v_3^\perp\| \cdots \|v_n^\perp\|$$

where

$$v_i^\perp \in \operatorname{Span}(v_1, \ldots, v_{i-1})^\perp.$$

Example: Compute the 3-volume of the parallelpiped defined by

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix}.$$

You Draw: (A picture of the parallelpiped defined by $v_1, v_2, v_3$)

<u>Want to compute:</u>

$$\|v_1\| \, \|v_2^{\perp}\| \, \|v_3^{\perp}\| = V(v_1, v_2, v_3)$$

$\|v_1\| = 2.$

$$v_2^\perp \in \underbrace{\mathrm{Span}(v_1)^\perp}_{y\text{-}z \text{ plane}}$$

$$v_2^\perp = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}. \quad \|v_2^\perp\| = 3.$$

$$v_3^\perp \in \mathrm{Span}(v_1, v_2)^\perp$$
$$= (x\text{-}y \text{ plane})^\perp$$
$$= z \text{ axis}.$$

$$v_3^\perp = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad \|v_3^\perp\| = 1.$$

$$\text{Volume}(v_1, v_2, v_3) = 2 \cdot 3 \cdot 1$$
$$= 6.$$

Thm: Let $A$ be an $n \times n$ matrix then $A$ is invertible if and only if the $n$-parallelpiped defined by the columns of $A$ has nonzero volume.

Why? $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

is invertible if and only if the columns $v_1, \ldots, v_n$ are linearly independent.

Consider:

$V(v_1, \ldots, v_n) = 0 \iff$

$\|v_1\| \, \|v_2^\perp\| \cdots \|v_n^\perp\| = 0$

$\iff \|v_i^\perp\| = 0$ for some $i$

$\iff v_i = v_i^{\parallel} \in \mathrm{Span}(v_1, \ldots, v_{i-1})$

$\iff v_i$ is redundant.

$V(v_1,\dots,v_n)=0 \iff$

$v_1,\dots,v_n$ are <u>NOT</u>

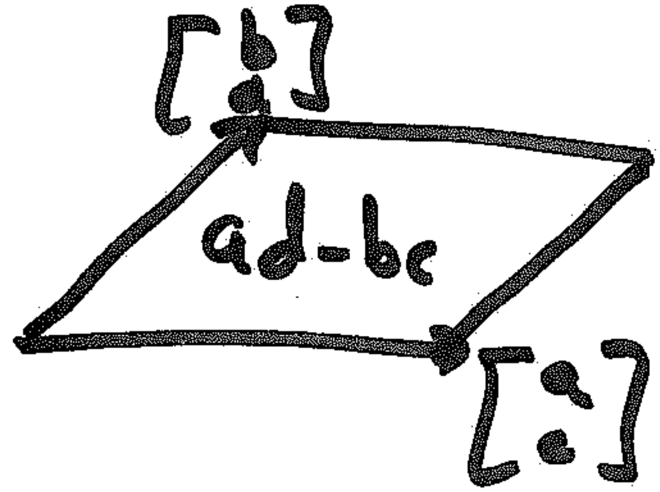
linearly independent. $\square$

The volume is a number which tells you if the matrix is invertible.

For a $2 \times 2$ matrix, we know another number which indicates if $A$ is invertible:

<u>the determinant</u>!

Thm: Let $A$ be a $2\times 2$ Matrix $|\det(A)|$ is the 2-volume of the image of the unit cube under $A$ (i.e. $|\det(A)|$ is the area of the parallelogram defined by the columns of $A$).

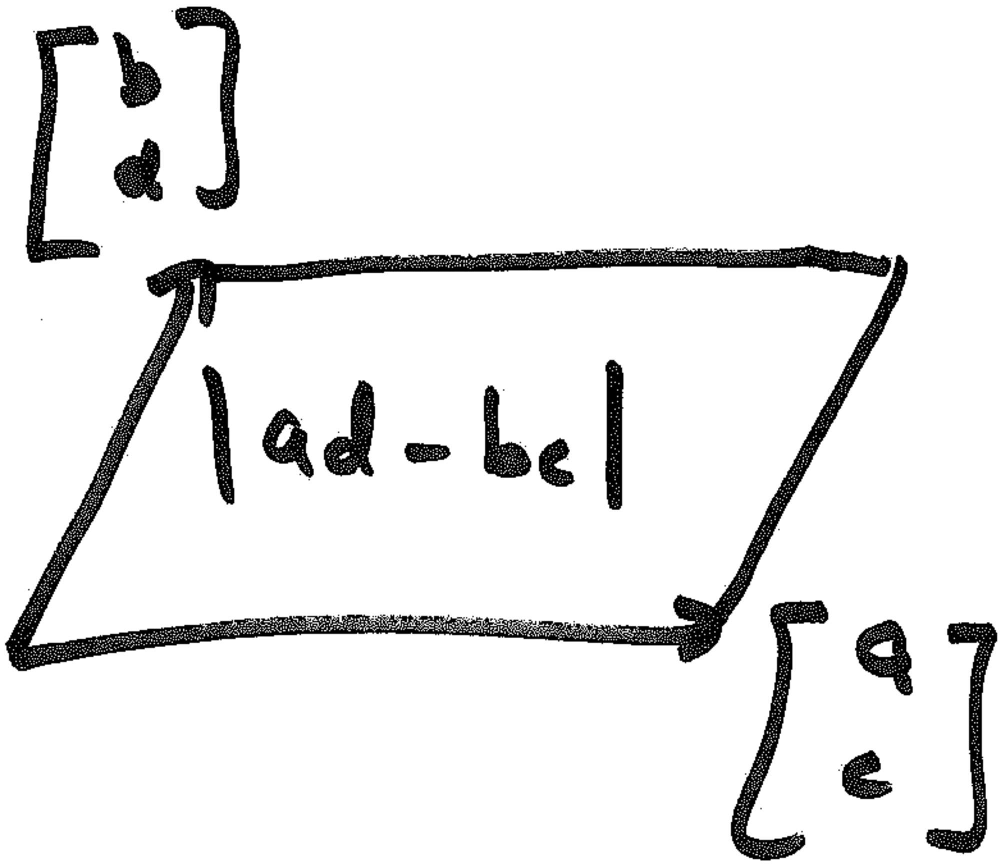


Thm: Let $A$ be a $2 \times 2$ matrix

$|\det(A)|$ is the 2-volume (area) of the image of the unit cube (square) under $A$.

(i.e. $|\mathrm{Det}(A)|$ is the area of the parallelogram defined by the columns of $A$).

i.e.


$\begin{bmatrix} b \\ d \end{bmatrix}$
$|ad - bc|$
$\begin{bmatrix} a \\ c \end{bmatrix}$

why is this true?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$v_1 = \begin{bmatrix} a \\ c \end{bmatrix} \quad (v_1 \neq \vec{0}).$$

$$v_2 = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Then $\mathrm{Span}(v_1)^{\perp}$ is spanned

by $\begin{bmatrix} -c \\ a \end{bmatrix}$.

$$v_2^{\perp} = \frac{v_2 \cdot \begin{bmatrix} -c \\ a \end{bmatrix}}{\begin{bmatrix} -c \\ a \end{bmatrix} \cdot \begin{bmatrix} -c \\ a \end{bmatrix}} \begin{bmatrix} -c \\ a \end{bmatrix}.$$

$$v_2 \cdot \begin{bmatrix} -c \\ a \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \cdot \begin{bmatrix} -c \\ a \end{bmatrix}$$

$$= ad - bc$$

$$= \det(A).$$

$$\begin{bmatrix} -c \\ a \end{bmatrix} \cdot \begin{bmatrix} -c \\ a \end{bmatrix} = a^2 + c^2$$

$$= \|v_1\|^2.$$

$$v_2^{\perp} = \frac{\det(A)}{\|v_1\|^2} \begin{bmatrix} -c \\ a \end{bmatrix}.$$

$$\|v_1\| \, \|v_2^{\perp}\| = \|v_1\| \frac{|\det(A)|}{\|v_1\|^2} \|v_1\|$$

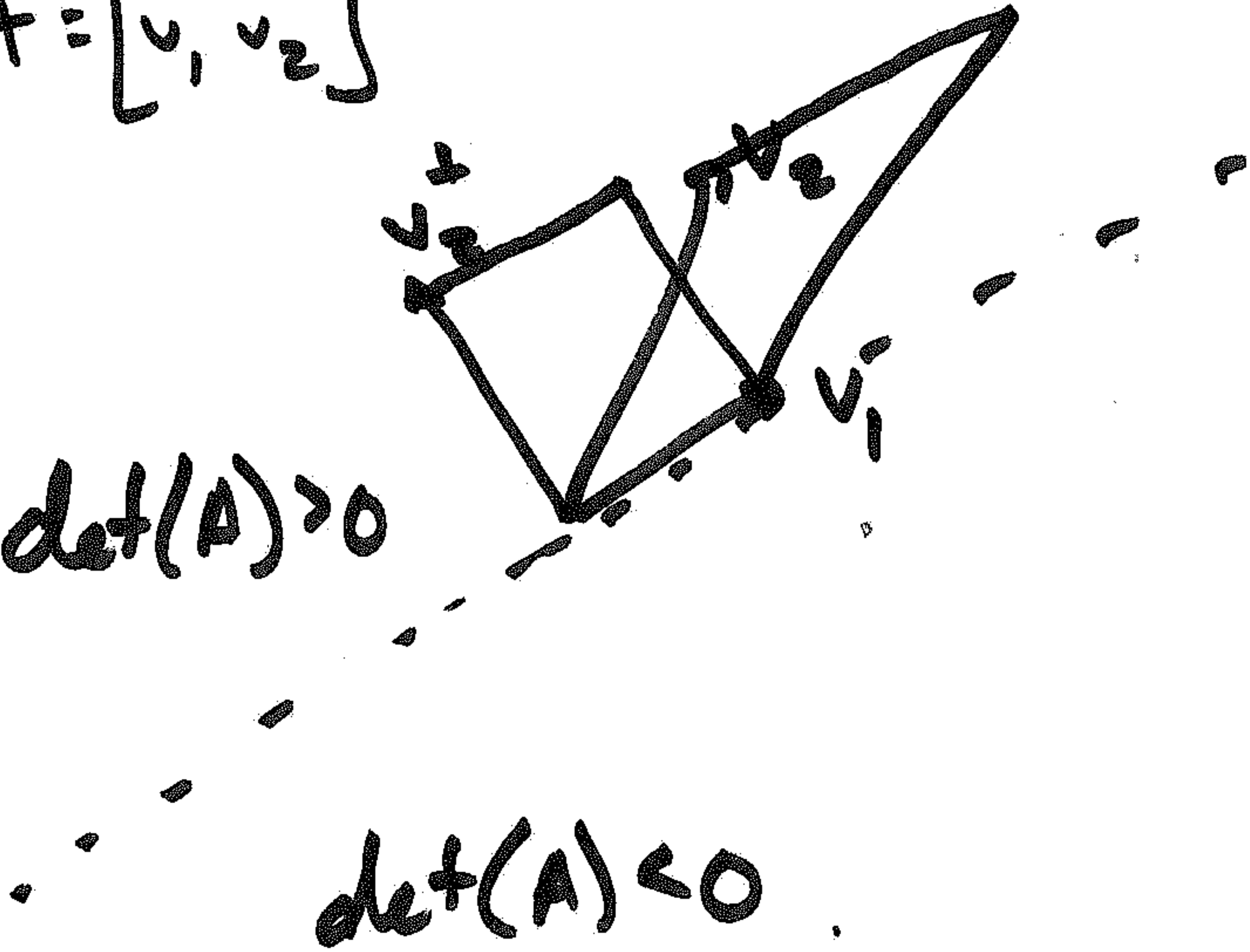
$$= |\det(A)|.$$

$$\boxed{V(v_1, v_2) = |\det(A)|.}$$

$\det(A) > 0$ if $v_2^{\perp}$ points in the same direction as $\begin{bmatrix} -c \\ a \end{bmatrix}$.

($\det(A) < 0$ if $v_2^{\perp}$ is opposite direction)

$A = [v_1, v_2]$




The sign of the determinant indicates the orientation of $v_2$ relative to $v_1$.

An algebraic property of the determinant:

If one fixes one column of a matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

and <u>varies</u> the other

e.g. $A = \begin{bmatrix} 3 & x \\ 7 & y \end{bmatrix}$

Then $\mathrm{Det}(A) = 3x + 7y$ is a linear function, (of that column)

Def/Thm: Let $\mathbb{R}^{n \times n}$ be the set of all $n \times n$ matrices. There is a unique function

$$\mathrm{Det}: \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$$

(1) $|\mathrm{Det}(A)| = V(v_1, \ldots, v_n)$

$v_1, \ldots, v_n$ are the columns of $A$.

(2) $\mathrm{Det}(I) = 1$.

(3) The determinant is a linear transformation in each column of $A$ i.e.

$$\mathrm{Det}\left(\begin{bmatrix} | & & | & | \\ v_1, & \dots & v_{n-1} & kv_n + v_n' \\ | & & | & | \end{bmatrix}\right)$$

$$= k\,\mathrm{Det}\begin{bmatrix} | & & | & | \\ v_1 & \dots & v_{n-1} & v_n \\ | & & | & | \end{bmatrix}$$

$$+ \mathrm{Det}\left(\begin{bmatrix} | & & | & | \\ v_1, & \dots, & v_{n-1} & v_n' \\ | & & | & | \end{bmatrix}\right).$$