Lecture 2: Solving Systems of Linear Equations (Part II).

Problem: Find all x, y, z such that

$$x + 2y + 3z = 39$$
$$x + 2y + 2z = 34$$
$$3x + 2y + z = 26.$$

Solution

Idea: These equations involve 9 non-zero terms on their left hand sides. We'll use algebra to systematically eliminate non-zero terms until we're left with the simplest equations describing the solution set.

I $\mapsto$ I
II $\mapsto$ II − I
III $\mapsto$ III − 3I

$$\begin{array}{ll}
\mathrm{I} & x + 2y + 3z = 39 \\
\mathrm{II} & x + 2y + 2z = 34 \\
\mathrm{III} & 3x + 2y + z = 36
\end{array}$$

$\xRightarrow{\ 3\ }$ Eliminate x-terms

I
$\xLeftarrow[\mathrm{III}+3\mathrm{I}]{\mathrm{II}+\mathrm{I}}$

$$\begin{aligned}
x + 2y + 3z &= 39 \\
-z &= -5 \\
-4y - 8z &= -91
\end{aligned}$$

III $\leftrightarrow$ II

$\Longleftrightarrow$ 1

$$\begin{aligned}
x + 2y + 3z &= 39 \\
-4y - 8z &= -91 \\
-z &= -5
\end{aligned}$$

I
$-\frac{1}{4}$ II
−1 III
$\Longrightarrow$ 2

$$\begin{aligned}
x + 2y + 3z &= 39 \\
y + 2z &= 22.75 \\
z &= 5
\end{aligned}$$

I
$\Longleftarrow$
−4 II
−1 III

I − 2II

I − 2II
II
III
$\Longrightarrow$ 3
Eliminate y-terms

$$\begin{aligned}
x + 0 - z &= -6.5 \\
y + 2z &= 22.75 \\
z &= 5
\end{aligned}$$

I + III
II − 2III
$\Longrightarrow$ 3
Eliminate z-terms

$$\begin{aligned}
x &= -1.5 \\
y &= 12.75 \\
z &= 5.
\end{aligned}$$

Algebraic Manipulation says that

$x = -1.5$

$y = 12.75$

$z = 5$

is the only possible solution.

Plugging into original equations we see that this is indeed a solution.

This manipulation (while correct) took a long time.

Writing down x, y, z and $+$ signs in each step is slowing us down.

~~Our first~~
We will introduce notation so that we don't have to do so much writing.

We can encode our system of linear equations as an augmented matrix:

$$\begin{aligned} x + 2y + 3z &= 39 \\ x + 2y + 2z &= 34 \\ 3x + 2y + z &= 26 \end{aligned} \iff \left[\begin{array}{ccc|c} 1 & 2 & 3 & 39 \\ 1 & 2 & 2 & 34 \\ 3 & 2 & 1 & 26 \end{array}\right]$$

Examples

$$\begin{aligned} 3x + 2y &= 7 \\ x + 5y &= 2 \end{aligned} \iff \left[\begin{array}{cc|c} 3 & 2 & 7 \\ 1 & 5 & 2 \end{array}\right].$$

$$x_1 + 3x_2 + 7x_4 + 8x_5 = 1 \iff \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 8 & 1 \end{array}\right]$$

x_3 is missing.

Is there an x_6 is missing?

$$X_1 + 3X_2 + 7X_4 + 8X_5 = 1$$

To write down augmented matrix we have to know how many variables there are and we should have a preferred ordering of our variables.

Algebraic manipulations of equations correspond to algebraic manipulations of rows of matrix.

Solution using aug. matrix:

(1) Encode system as augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 39 \\ 1 & 2 & 2 & 34 \\ 3 & 2 & 1 & 36 \end{array}\right]$$

(2) Preform row operations to increase # of zeros of left of dotted line.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 39 \\ 1 & 2 & 2 & 34 \\ 3 & 2 & 1 & 26 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 39 \\ 0 & 0 & -1 & -5 \\ 0 & -4 & -8 & -91 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 39 \\ 0 & -4 & -8 & -91 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 39 \\ 0 & 1 & 2 & 22.75 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -6.5 \\ 0 & 1 & 2 & 22.75 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1.5 \\ 0 & 1 & 0 & 12.75 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x = -1.5$, $y = 12.75$, $z = 5$.

When Solving a System of Linear equations which manipulations of the rows do we need / can we use?

Something you shouldn't do

Starting with an equation

$$y + 3x = 7 \xrightarrow[\text{it by } 0]{\text{Multiply}} 0 \cdot y + 0 \cdot x = 0$$

or

$$0 = 0$$

Solutions are a line

Solutions are all x and y

* This algebraic manipulation increases the set of solutions.

* In order to make sure our Solution Set stays the Same we want to make manipulations that are reverseable.

To solve systems of linear equations (in matrix form) its okay / enough to use the following <u>row operations</u>:

<u>Elementary Row Operations</u>

(1) Switch two rows of your augmented matrix $R_i \longleftrightarrow R_j$

(2) Multiply a row by a <u>non-zero</u> number $R_i \longmapsto cR_i$

(3) Add a multiple of the i^{th} row to the j^{th} row

$$R_j \longmapsto R_j + cR_i$$

Problem: Solve

$$9z = 3$$
$$8x + 4y + 3z = 25$$
$$2x + y + 6z = 8.$$

Solution

Step 1: encode as augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 0 & 9 & 3 \\ 8 & 4 & 3 & 25 \\ 2 & 1 & 6 & 8 \end{array}\right] \xRightarrow{\mathrm{I} \leftrightarrow \mathrm{III}} \left[\begin{array}{ccc|c} 2 & 1 & 6 & 8 \\ 8 & 4 & 3 & 25 \\ 0 & 0 & 9 & 3 \end{array}\right]$$

Step 2: Apply row operations to decrease complexity.

$$\xRightarrow{\mathrm{II} - 4\mathrm{I}} \left[\begin{array}{ccc|c} 2 & 1 & 6 & 8 \\ 0 & 0 & -21 & -7 \\ 0 & 0 & 9 & 3 \end{array}\right]$$

$$\underset{\frac{1}{9}\mathrm{III}}{\xRightarrow{-\frac{1}{21} \cdot \mathrm{II}}} \left[\begin{array}{ccc|c} 2 & 1 & 6 & 8 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right].$$

$$\underset{\mathrm{III} - \mathrm{II}}{\xRightarrow{\mathrm{I} - 6\mathrm{II}}} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Decoding we have that the system

$$2x + y = 6$$
$$z = 1/3$$

has the same solution set.

Next time we'll start by understanding this set geometrically.

$$\xrightarrow[\text{III} - \text{II}]{\text{I} - 6\,\text{II}} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

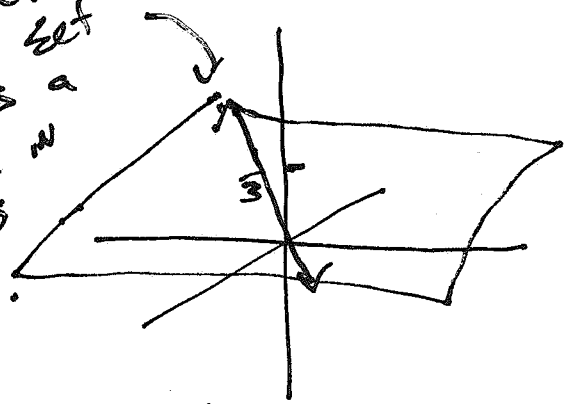
I claim at this point there are no more elementary row operations we can do to make the matrix have more zero-entries.

Plane $2x + y = 6$

Plane $z = 1/3$

$$\Longrightarrow \boxed{\begin{array}{l} x + \frac{y}{2} = 6 \\ z = 1/3 \end{array}}.$$

Solution set is a line in this plane.



Solution set is contained in the plane $z = 1/3$

∞ - many solutions.