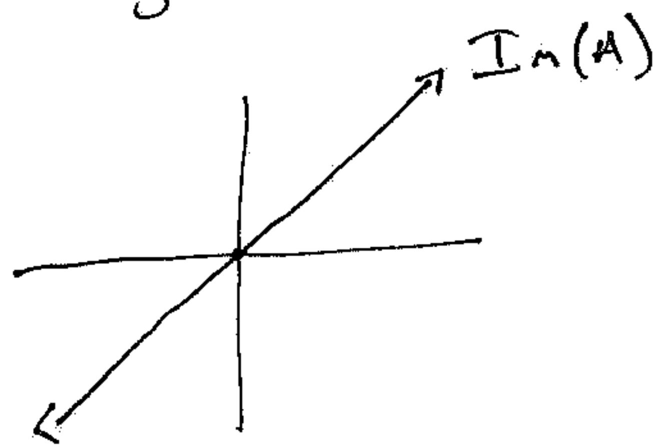
Lecture 18: Approximate Solutions to $Ax = b$.

Today, we'll talk about real world <u>complications</u> that arise when solving a matrix equation

$$Ax = b.$$

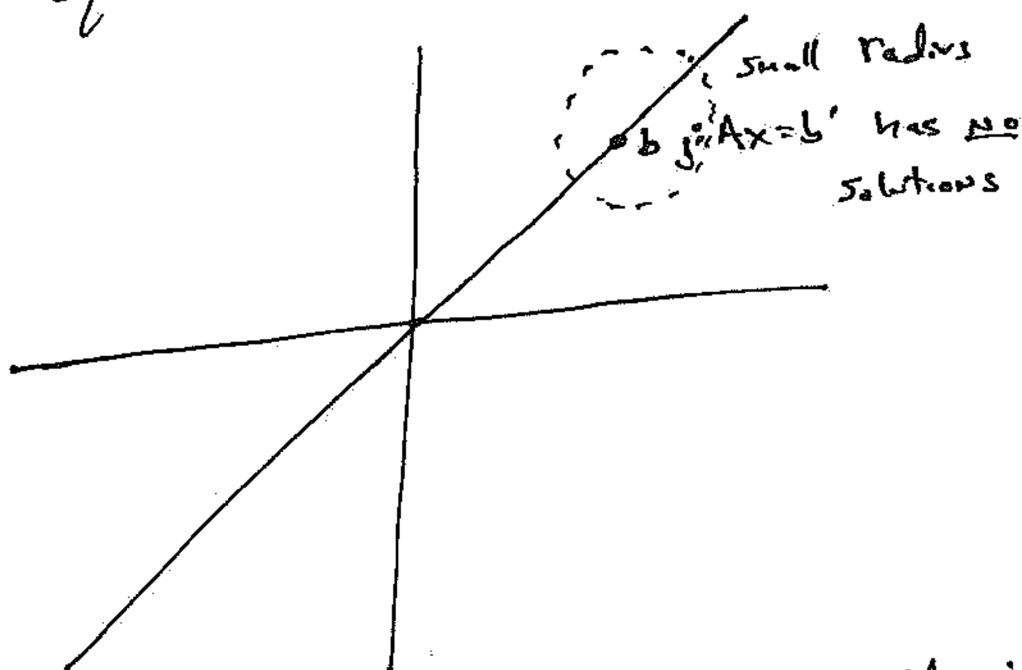
Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

We can solve $Ax = b$ if $b \in \mathrm{Im}(A)$, which in this case is the line of slope 1 in $\mathbb{R}^2$ through the origin.



We can't solve $Ax=b$ if b is <u>not</u> on this line (i.e. $b \notin \mathrm{Im}(A)$).

<u>Here's the issue</u>: Starting with a vector $b \in \mathrm{Im}(A)$ <u>Most</u> small pertubations of b will lie outside the image of A, i.e. for most b' close to b the equation $Ax=b'$ has <u>NO</u> solutions



On the other hand, in the real world it's hard to be <u>exact</u>, and small <u>pertubations</u> of your equations naturally arise (e.g. coming from noise in data, round off errors, etc.)

So even though you might want to solve $Ax = b$ (in theory), in practice you'll only be able to ask for solutions to

$Ax = b'$ for b' close to b (and there are none!)

To cope with this issue, we modify our fundamental problem: solving $Ax = b$.

$$\boxed{\text{General Problem: Given an } n \times m\text{-matrix } A \text{ and vector } b \in \mathbb{R}^n\text{, find a vector } x \in \mathbb{R}^m \text{ which minimizes the distance between } Ax \text{ and } b.}$$

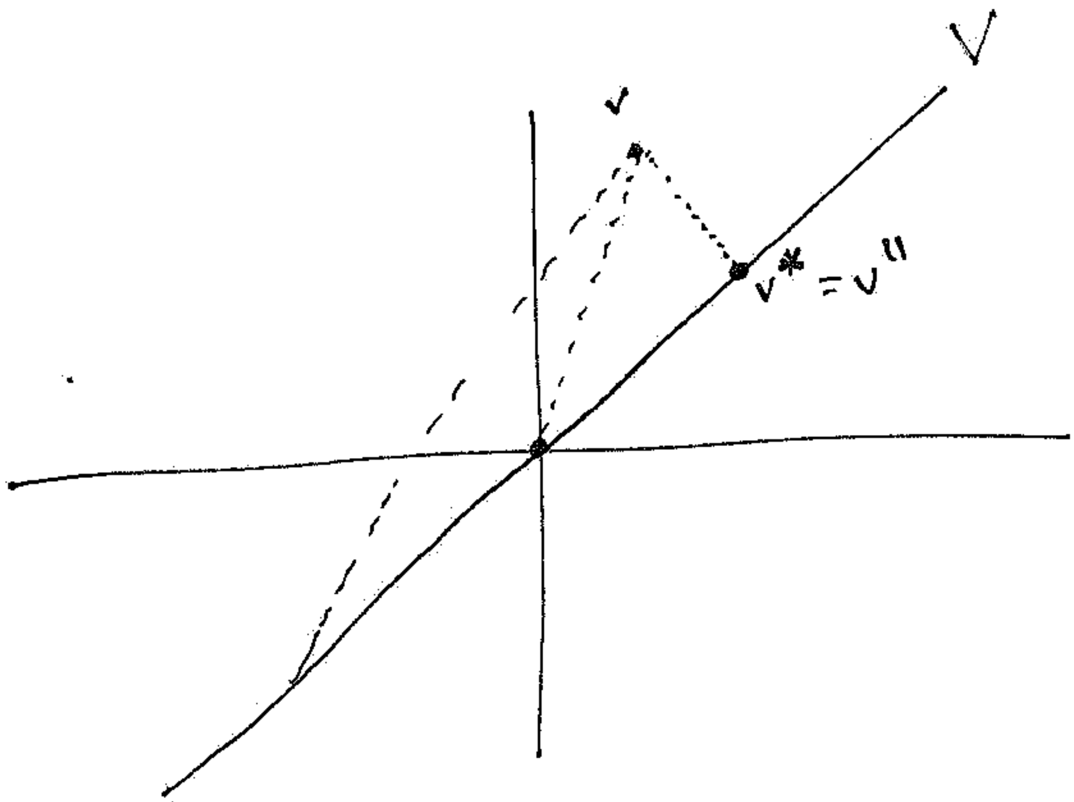
~~(Note the vector x^* is a solution to $Ax = b^*$ where b^*~~

When $A\vec{x} = b$ has a solution, then the vectors which minimize the distance between $Ax \in \mathrm{Im}(A)$ and b are simply the solutions to $Ax = b$. More generally, the vectors which minimize the distance are solutions to $Ax = b^*$ where b^* is the closest vector in the image of A to b.

What is b^*?

Thm: Let $v \in \mathbb{R}^n$ be a vector and $V \subseteq \mathbb{R}^n$ be a subspace. The closest vector to v is $v^{\parallel}$.

Example:



The distances between a vector v and various points in the subspace V. Note that the distance is minimized for

$$v^* = v^{\parallel}.$$

Def: An <u>approximate</u> (or <u>least squares</u>) solution to $Ax = b$ is a solution to $Ax = b^{\parallel}$ where $b^{\parallel} \in \mathrm{Im}(A)$ and $b = b^{\parallel} + b^{\perp}$ where $b^{\perp} \in \mathrm{Im}(A)^{\perp}$. The quantity $\| Ax - b \| = \| b^{\perp} \|$ is called the <u>error</u> of the approximation.

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1.1 \end{bmatrix}$.

Then $\mathrm{Im}(A) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

(1) $$b^{\parallel} = \frac{b \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2.1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.05 \\ 1.05 \end{bmatrix}.$$

(2) Since $x = \frac{1}{2} \begin{bmatrix} 1.05 \\ 1.05 \end{bmatrix}$ is a solution to $Ax = \begin{bmatrix} 1.05 \\ 1.05 \end{bmatrix}$, we have $\frac{1}{2} \begin{bmatrix} 1.05 \\ 1.05 \end{bmatrix}$ is an <u>approximate</u> solution to $Ax = b$.

In practice, we'll compute approximate solutions using the following theorem.

$\boxed{\text{Thm: The approximate solutions to } Ax = b \text{ are the solutions to the (consistent) matrix equation } A^T A x = A^T b.}$

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A^T = A$, $b = \begin{bmatrix} 1 \\ 1.1 \end{bmatrix}$.

$$A^T b = \begin{bmatrix} 2.1 \\ 2.1 \end{bmatrix}, \qquad A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 2 & 2 & 2.1 \\ 2 & 2 & 2.1 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{cc|c} 1 & 1 & 1.05 \\ 0 & 0 & 0 \end{array}\right]$$

Solutions are

$$\begin{bmatrix} 1.05 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1.05 \\ -1.05 \end{bmatrix}$$

$t = -1/2$ is the solution found before.

Why is this theorem true?

(1) $Ax = b^{\parallel}$ if and only if $b^{\perp} = b - Ax$,
i.e. $(b - Ax) \cdot v = 0$ for all $v \in Im(A)$

(2) To check $(b - Ax) \cdot v = 0$ for all $v \in im(A)$, it suffices to check $(b - Ax) \cdot v_i = 0$ for all columns of A (since every $v \in Im(A)$ can be written as $c_1 v_1 + c_2 v_2 + \dots + c_m v_m$

and

$$(b - Ax) \cdot (c_1 v_1 + \dots + c_m v_m) = c_1 (b - Ax) \cdot v_1 + \dots + c_m (b - Ax) \cdot v_m)$$

(3) Using Trick from lecture 17, we have $(b - Ax) \cdot v_i = 0$ for all columns v_i of A
iff $A^T (b - Ax) = 0 \iff A^T b = A^T A x$.

Steps 2 + 3 are worth stating as a theorem:

Thm: $\ker(A^T) = \operatorname{im}(A)^\perp$.

Example: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Image is line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Kernel is the plane orthogonal to this line, e.g. the solutions to

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z = 0$$

Let's do some problems computing approximate solutions.

Problem 1: Find least-squares (approximate) solution to $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$A^T b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^T A x = A^T b \implies Ix = A^T b$$

$$\implies x = A^T b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem:

Find least squares solution to $Ax=b$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Solution:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}.$$

$$A^T A x = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 6 \\ 6 \end{bmatrix}.$$

Row reduce

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 6 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 1 & 6 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & -3 & -6 \end{array}\right].$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$$x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Alternately: Since $\det(A^TA) = 4 - 1 = 3 \neq 0$

$$x = (A^TA)^{-1}A^Tb$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$