Lecture 17

Orthogonal Transformations

Def: A linear transformation

$$T: \mathbb{R}^n \to \mathbb{R}^n$$

is called <u>orthogonal</u> if

$$(Tx)\cdot(Ty) = x \cdot y$$

for all $x, y \in \mathbb{R}^n$.

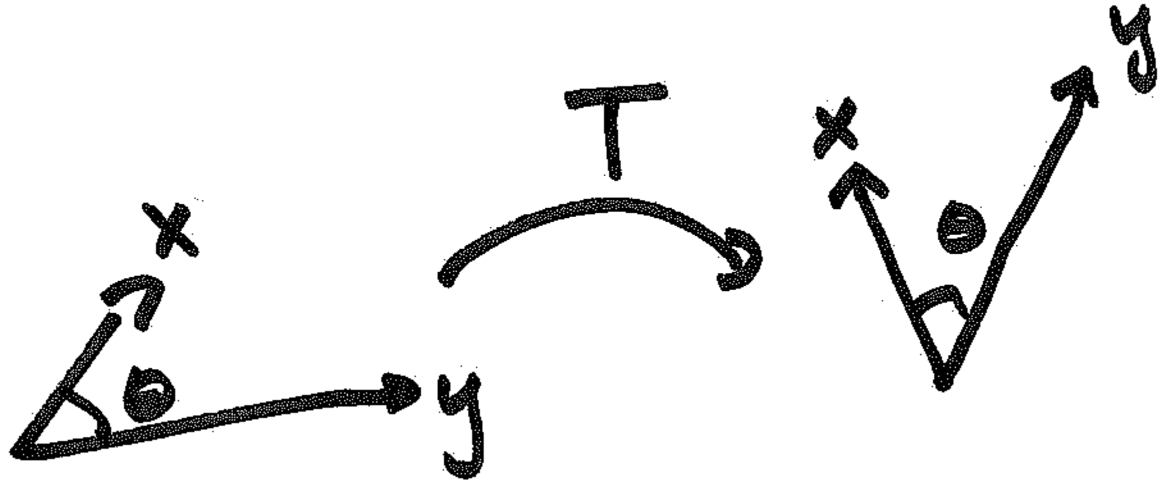
(i.e. T preserves dot products)

Geometrically, the dot product encodes lengths of vectors and the angles between them.

Orthogonal transformations preserve lengths and angles.

$$\| Tx \| = \| x \|$$

$$\theta(Tx, Ty) = \theta(x, y).$$


x
θ
y
T
x
θ
y

Examples:

(0) identity Matrix

(1) Rotation in $\mathbb{R}^2$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(2) Reflection over a line containing $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

(3) (New) Reflection through a subspace $V \subseteq \mathbb{R}^n$.

This map is defined by

$$\mathrm{Ref}_V(w) = w^{\parallel} - w^{\perp}$$

$w^{\parallel} \in V$ and $w^{\perp} \in V^{\perp}$.

Why is Ref_V orthogonal?

Check

$$\mathrm{Ref}_V(x) \cdot \mathrm{Ref}_V(y) = x \cdot y.$$

$$(x^{\parallel} - x^{\perp}) \cdot (y^{\parallel} - y^{\perp})$$

$$= x^{\parallel} \cdot y^{\parallel} - \cancelto{0}{x^{\perp} \cdot y^{\parallel}} - \cancelto{0}{x^{\parallel} y^{\perp}} + x^{\perp} y^{\perp}.$$

$$= x^{\parallel} \cdot y^{\parallel} + x^{\perp} y^{\perp}.$$

($x^{\perp} \cdot y^{\parallel} = 0$ because $x^{\perp}$ is perp to V and $y^{\parallel} \in V$).

$$x \cdot y = (x^{\parallel} + x^{\perp})(y^{\parallel} + y^{\perp})$$

$$= x^{\parallel} y^{\parallel} + \cancelto{0}{x^{\perp} y^{\parallel}} + \cancelto{0}{y^{\perp} x^{\parallel}} + y^{\perp} x^{\perp}$$

$$= x^{\parallel} y^{\parallel} + x^{\perp} y^{\perp}.$$

These are the same, so

Ref_v preserves dot products

$\Longrightarrow$ orthogonal.

Non-example

Orthogonal Projection is <u>NOT</u> orthogonal.

Why? $w \in V^{\perp}$ $w \neq 0$

then

$$\mathrm{Proj}_V(w) = 0$$

so

$$\|\mathrm{Proj}_V(w)\| \neq \|w\|.$$

How can you recognize if a matrix encodes an orthogonal transformation?

Thm: The matrix

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

is orthogonal if and only if $v_1, \ldots, v_n$ is an orthonormal basis for $\mathbb{R}^n$, i.e. $v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

Example

Rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\qquad\qquad\qquad v_1 \qquad\quad v_2$

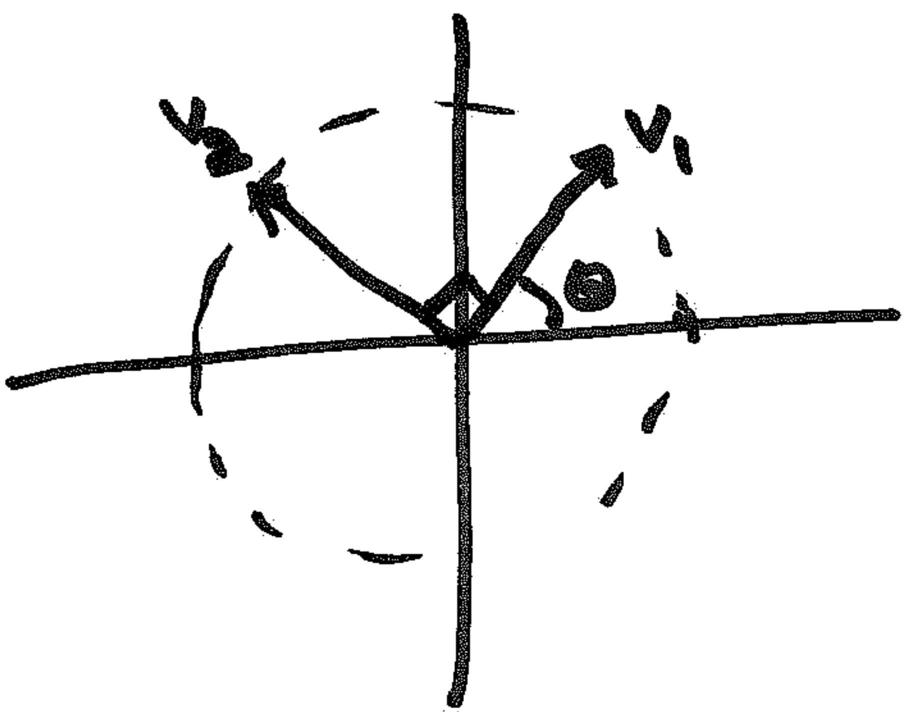
To check if this matrix is orthogonal we compute

$$v_1 \cdot v_1 = \cos^2\theta + \sin^2\theta = 1.$$

$$v_1 \cdot v_2 = -\cos\theta \sin\theta + \cos\theta \sin\theta = 0$$

$$v_2 \cdot v_2 = (-\sin\theta)^2 + \cos^2\theta = 1.$$

Geometrically



Why is this theorem true?

The columns $v_1, \ldots, v_n$ of A are images of the standard basis vectors $e_1, \ldots, e_N$ under A, i.e

$$v_i = Ae_i$$

Since $e_1, \ldots, e_N$ is an orthonormal basis, we have if A is orthonormal then $Ae_1, \ldots, Ae_N$ is an orthonormal basis.

Going the other direction is a bunch of algebra along the lines of example 3. (I won't do this). $\square$

Problem: Is

$$A = \frac{1}{3} \begin{array}{c} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \\ \begin{matrix} v_1 & \quad v_2 & \quad v_3 \end{matrix} \end{array}$$

orthogonal?

Solution

We have to check if the columns of A form an orthonormal basis.

Check $\quad v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$$v_1 \cdot v_1 = \left(\frac{1}{3}\right)^2 (1\cdot 1 + 2\cdot 2 + 2\cdot 2) = 1$$

$$v_1 \cdot v_2 = \left(\frac{1}{3}\right)^2 (1\cdot(-2) + 2(-1) + 2\cdot 2) = 0$$

$$v_1 \cdot v_3 = 0$$

$$v_2 \cdot v_3 = 0$$

$$v_2 \cdot v_2 = 1$$

$$v_3 \cdot v_3 = 1$$

$\Longrightarrow$ v_1, v_2, v_3 are an orthonormal basis $\Longrightarrow$ A is orthogonal!

It's annoying to list all these combinations to compute $v_i \cdot v_j$.

Trick to avoid this:

Note for vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underbrace{x \cdot y}_{\text{dot product}} = \underbrace{\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\text{Matrix mult.}}$$

When we multiply a pair of matrices BA we compute the <u>dot product</u> of every <u>row</u> of B with every <u>column</u> of A.

Consider the matrix A^T obtained from A by switching rows + columns.

$$A^T = \begin{array}{c} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix} \end{array} \begin{array}{l} v_1 \\ v_2 \\ v_3 \end{array}$$

Compute

$$A^T A = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\ v_1 \cdot v_2 & v_2 \cdot v_2 & v_2 \cdot v_3 \\ v_1 \cdot v_3 & v_3 \cdot v_2 & v_3 \cdot v_3 \end{bmatrix}$$

$$\stackrel{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I.$$

$$A^T A = \left(\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}\right)\left(\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 9/9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Yes v_1, v_2, v_3 is

an orthonormal basis.

Def: Let A be an $n \times m$ matrix. The **transpose** of A is the $m \times n$ matrix obtained by ~~sotti~~ switching the ij-th entry with the ji-th entry (equiv. switching rows and columns)

Examples:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Thm: If $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

then

$$A^T A = \begin{bmatrix} v_1 \cdot v_1 & v_2 \cdot v_1 & \cdots \\ v_1 \cdot v_2 & v_2 \cdot v_2 & \\ \vdots & & \ddots \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} v_i \cdot v_j \end{bmatrix}}_{ij\text{th entry}}$$

Thm: A linear transformation

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is orthogonal if and only if

$$A^T A = I$$

i.e. A is invertible and

$$A^{-1} = A^T.$$

Example

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T$$

inverse

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta \end{bmatrix}$$

rotating θ-degrees clockwise

Thm: The set of all orthogonal matrices is closed under inversion and matrix multiplication, i.e. if A, B are orthogonal then

① AB is orthogonal

② $A^T = A^{-1}$ is orthogonal.

why? If A, B preserve lengths and angles then AB preserves lengths and angles

Same for A^{-1}!

Example:

$$A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}.$$

$$A^2 = \frac{1}{9}\begin{bmatrix} 1 & 4 & 8 \\ -4 & -7 & 4 \\ 8 & -4 & 1 \end{bmatrix}$$

$$A^3 = \frac{1}{27}\begin{bmatrix} 25 & 10 & 2 \\ -10 & 23 & 10 \\ 2 & -10 & 25 \end{bmatrix}$$

are all orthogonal.

$$1^2 + 2^2 + 2^2 = 3^2.$$