Lecture 16

Projections and Ortho-normal Bases

Problem: Compute the projection of the vector

$$w = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

onto the subspace $V \subseteq \mathbb{R}^4$

spanned by

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

What are we trying to do?

$$w = w^{\parallel} + w^{\perp}$$

where $w^{\parallel} \in V$ and $w^{\perp} \cdot v = 0$ for all $v \in V$

$$\boxed{\text{Goal: Find } w^{\parallel}}$$

Idea:

$$w^{\parallel} = c_1 v_1 + c_2 v_2$$

So

$$w = \overbrace{c_1 v_1 + c_2 v_2}^{w^{\parallel}} + w^{\perp}$$

Consider

Numbers

$$w \cdot v_1 = c_1 (v_1 \cdot v_1) + c_2 (v_2 \cdot v_1) + \cancelto{0}{w^{\perp} \cdot v_1}$$

$$w \cdot v_2 = c_2 (v_1 \cdot v_2) + c_2 (v_2 \cdot v_2)$$

i.e. Since

$v_1 \cdot v_1 = 5 \quad v_2 \cdot v_2 = 3$

$v_1 \cdot v_2 = 3$

$w \cdot v_1 = 3 \quad w \cdot v_2 = 2$

System of linear equations:

$$3 = c_1 5 + c_2 3$$

$$2 = c_1 3 + c_2 3$$

☆ By taking dot products with basis vectors we obtain a system of linear equations; we know how to solve those!

Solve system

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$A \quad x \quad = b.$$

check if A is invertible

$$\det(A) = 5 \cdot 3 - 3 \cdot 3 = 6 > 0.$$

So A^{-1} exists.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/6 \end{bmatrix}.$$

Unpacking

$$w'' = c_1 v_1 + c_2 v_2$$

$$= \frac{1}{2} v_1 + \frac{1}{6} v_2$$

$$\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} v_1 \cdot v_1 & v_2 \cdot v_1 \\ v_1 \cdot v_2 & v_2 \cdot v_2 \end{bmatrix}^{-1} \begin{bmatrix} w \cdot v_1 \\ w \cdot v_2 \end{bmatrix} \right)$$

$$w'' = \frac{1}{2} v_1 + \frac{1}{6} v_2$$

$$= \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 0 \\ 1/6 \\ 4/6 \end{bmatrix}$$

Check our math by computing

$$w^{\perp} = w - w^{\parallel}$$

and seeing

$$w^{\perp} \cdot v_1 = 0$$
$$w^{\perp} \cdot v_2 = 0.$$

So

$$w^{\perp} = w - w^{\parallel}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 7/6 \\ 0 \\ 1/6 \\ 4/6 \end{bmatrix}$$

$$w^{\perp} = \begin{bmatrix} -1/6 \\ 1 \\ -1/6 \\ 1/3 \end{bmatrix}.$$

$$w^{\perp} \cdot v_1 = -2/6 + 1/3 = 0 \checkmark$$

$$w^{\perp} \cdot v_2 = -1/6 - 1/6 + 1/3$$
$$= 0 \checkmark$$

Thm: Let V be a subspace with basis $v_1, \dots, v_K$. Then

$$\text{Proj}_V(w) = c_1 v_1 + c_2 v_2 + \dots + c_K v_K$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_K \end{bmatrix} = \begin{bmatrix} v_1 \cdot v_1 & v_2 \cdot v_1 & \cdots \\ v_1 \cdot v_2 & v_2 \cdot v_2 & \\ \vdots & & \ddots \\ \vdots & & \end{bmatrix}^{-1} \begin{bmatrix} w \cdot v_1 \\ w \cdot v_2 \\ \vdots \\ \vdots \\ w \cdot v_K \end{bmatrix}$$

ij^{th} entry is $v_i \cdot v_j$

$$\left[\begin{array}{c} v_i \cdot v_j \\ \\ \end{array}\right]$$

ij^{th} entry

Project onto line spanned by v

$$w^{\parallel} = \frac{v \cdot w}{v \cdot v}$$

Computing the projection will be easier / harder depending on how easy / hard it is to invert the matrix of dot products.

It's simple when this matrix is the identity.

$$\begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \\ v_3 \cdot v_1 & & \ddots \\ \vdots & & \end{bmatrix} = I$$

when

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j . \end{cases}$$

i.e.

v_i is a unit vector

and $v_i \perp v_j$ if $i \neq j$.

Def: Let $V \subseteq \mathbb{R}^n$ be a subspace with basis $u_1, \ldots, u_k$. We say $u_1, \ldots, u_k$ is an <u>orthonormal basis</u> if

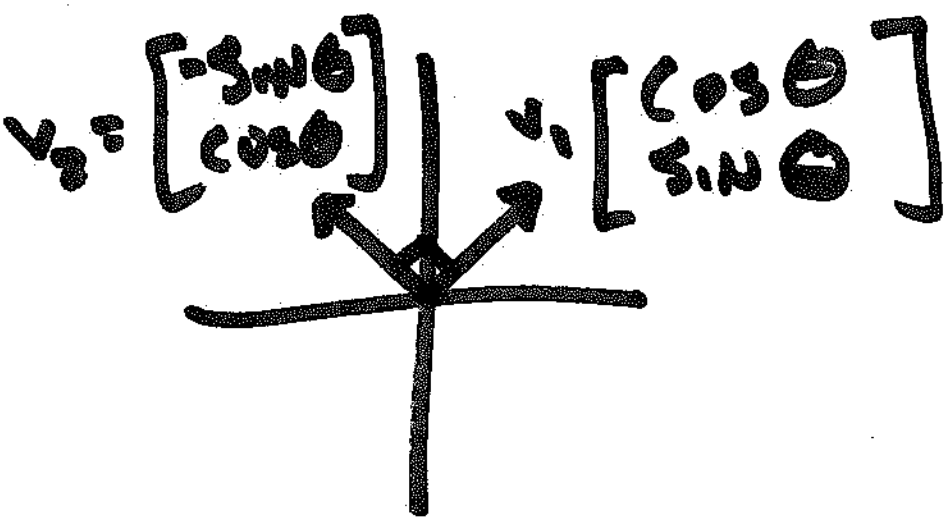
u_i is a unit vector for all i and $u_i \perp u_j$ if $i \neq j$.

Examples:

$V = \mathbb{R}^n \quad e_1, \ldots, e_n$

is an orthonormal basis.

$V = \mathbb{R}^2$



Why do we care about orthonormal bases?

The projection formula is <u>easy</u>.

Thm: If V is a subspace with ortho. basis $u_1, \ldots, u_k$ then

$$\mathrm{Proj}_V(w) = c_1 u_1 + \ldots + c_k u_k$$

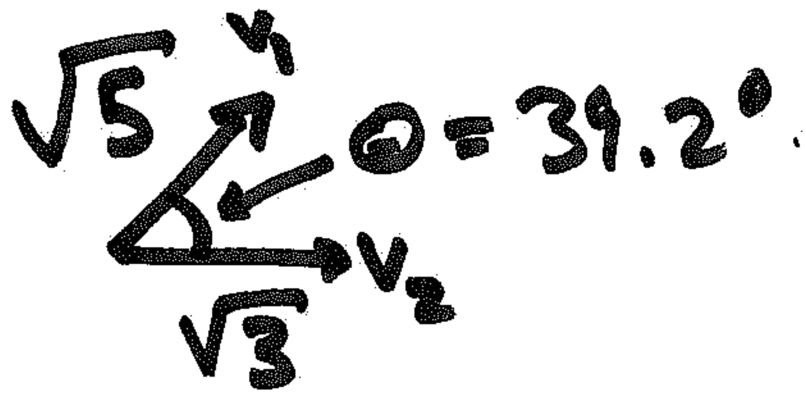
where

$$c_i = w \cdot u_i .$$

Example: $V = \text{Span}\left(\underset{v_1}{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} \right)$

Find orthonormal basis for V.

Starting basis



$\|v\| = v \cdot v$

$v_1 \cdot v_2 = \|v_1\| \|v_2\| \cos\theta$

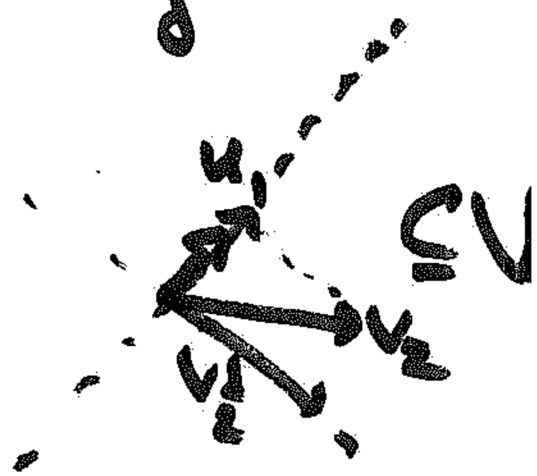
$\frac{3}{\sqrt{15}} = \cos\theta \implies \text{Arccos}\left(\frac{3}{\sqrt{15}}\right)$

$= \theta = 39.2°$

Adjust vectors to produce orthonormal basis.

Step 1: Replace v_1 by

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



Step 2: Write

$$v_2 = v_2^{\parallel} + v_2^{\perp}$$

where $v_2^{\parallel} \in \mathrm{Span}(u_1)$

Replace v_2 by $v_2^{\perp}$.

$$v_2^{\perp} = v_2 - v_2^{\parallel}$$

$$= v_2 - (c_1) u_1$$

$$= v_2 - (u_1 \cdot v_2) u_1$$

Projection Formula.

$\subseteq V$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{\sqrt{5}} \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 0 \\ 1/\sqrt{5} \end{bmatrix}.$$

$$= \begin{bmatrix} -1/5 \\ 0 \\ \cancel{0} \\ 2/5 \end{bmatrix} \leftarrow 1 .$$

$$u_2 = \frac{v_2^\perp}{\|v_2^\perp\|} = \frac{5}{\sqrt{28}} \begin{bmatrix} -1/5 \\ 0 \\ 1 \\ 2/5 \end{bmatrix}.$$

$$\|v_2^\perp\| = \frac{\sqrt{28}}{5}$$

Answer:

$$\boxed{u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \frac{5}{\sqrt{28}} \begin{bmatrix} -1/5 \\ 0 \\ 1 \\ 2/5 \end{bmatrix}}$$

In general, this process of adjusting a basis to produce an orthonormal basis is called the Gram-Schmidt process.

It can be described as:

Starting with

$v_1, \ldots, v_k$ a basis

For each v_j consider

$$v_j = v_j^{\parallel} + v_j^{\perp}$$

with respect to

$$\mathrm{Span}(v_1, \ldots, v_{j-1})$$

then

$$u_j = \frac{v_j^{\perp}}{\|v_j^{\perp}\|}$$

$$v_j^{\perp} = v_j - v_j^{\parallel}$$

$$= v_j - \sum_{i=1}^{j-1} (u_i \cdot v_j) \cdot u_i$$

is an orthonormal basis.