Lecture 15

Orthogonality + Projections in \( \mathbb{R}^n \)
§ 1: Geometric Interpretation of dot products (revisited).

Thm: \( \mathbf{v} \cdot \mathbf{w} = 0 \) if and only if \( \mathbf{v} \) and \( \mathbf{w} \) are perpendicular.
Def: The length of a vector $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ is

$$\|v\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{v \cdot v}.$$ 

So $v \cdot v = \|v\|^2$. 

$$v \cdot v = 0 \iff \|v\| = 0 \iff v = 0.$$
Example:

\[
\sqrt{2^2 + 1^2} = \sqrt{5}
\]

\[
\|v\| = \sqrt{5}.
\]

Remark: Our definition of length is the Pythagorean theorem.
Def: We call $v \in \mathbb{R}^n$ a unit vector if
\[ \|v\| = 1. \]

Thm: If $v$ is non-zero, $\frac{v}{\|v\|}$ is a unit vector.

Circle radius 1
Thm: Let \( v_1, v_2 \in \mathbb{R}^n \)
then
\[
v_1 \cdot v_2 = \| v_1 \| \| v_2 \| \cos \theta
\]
where \( \theta \) is the angle between \( v_1 \) and \( v_2 \)

where \( 0 \leq \theta < \pi \)
Why is this formula true?

It relates two ways of describing projection onto a line.
So:

\[ v_1'' = \left( \frac{11 v_{11}}{11 v_{22}} \cos \theta \right) v_2 \]

On the other hand, you know

\[ v_1'' = \left( \frac{v_1 \cdot v_2}{v_2 \cdot v_2} \right) v_2 \]

Equating these:

\[ \frac{v_1 \cdot v_2}{11 v_{22}} = \frac{11 v_{11}}{11 v_{22}} \cos \theta \]

So

\[ v_1 \cdot v_2 = 11 v_{11} 11 v_{22} \cos \theta. \]
Problem: What is the angle between $v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$?

Using formula solve for $\theta$. 
\[ \mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{1}{2} \]

\[ ||\mathbf{v}_1|| = \sqrt{3/4} = \frac{\sqrt{3}}{2} \]

\[ ||\mathbf{v}_2|| = 1 \]

\[ \frac{1}{2} = \frac{\sqrt{3}}{2} \cos \theta \]

\[ \theta = \arccos \left( \frac{1}{\sqrt{3}} \right) \approx 54.7 \text{ degrees} \]
§ 2. Projection onto a subspace $S^+$ or the gonal complements.

Thm: Let $V \subseteq \mathbb{R}^n$ be a subspace, every vector $w \in \mathbb{R}^n$ can be written as 

$$w = w^\parallel + w^\perp$$

where $w^\parallel \in V$ and $w^\perp \in (V)^\perp$, for all $\vec{v} \in V$ ($w^\perp$ is perp. to all $\vec{v} \in V$).
Thm / Def: The map \( \text{Proj}_V : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by \( w \mapsto \overline{w} \) is called an orthogonal projection onto \( V \). \( \text{Proj}_V \) is a linear transformation.
Properties:

1. $\text{Proj}(\vec{v}) = \vec{v}$ if $\vec{v} \in V$.

2. $\text{Im}(\text{Proj}_v) = V$.

3. $w \in \text{ker}(\text{Proj}_v)$ if $w^\perp = 0$.
   
   So if $w = w^\perp$, i.e., $w$ is perpendicular to all $\vec{v} \in V$. 
Def: The orthogonal complement of a subspace $V$ is

$$V^\perp = \{ w \in \mathbb{R}^n : \forall v \in V, \ w \cdot v = 0 \}.$$
Examples of $V^\perp$

1. In $\mathbb{R}^2$, if $V$ is a line through $0$ of slope $m$, then $V^\perp$ is perpendicular line $v_1$ (slope $-\frac{1}{m}$).
2. In $\mathbb{R}^n$, if $V$ is a plane through $0$ then $V^\perp$ is the perpendicular line and v.v.

3. $(\mathbb{R}^n)^\perp = \mathbb{E}^3$.
   $\mathbb{E}^3^\perp = \mathbb{R}^n$. 
Properties of $V^\perp$

1. $V^\perp$ is a subspace (it's the kernel of the linear trans. $\text{Proj}_V$)

2. $(V^\perp)^\perp = V$

3. $\dim(V^\perp) = n - \dim(V)$

(Rank nullity: $\dim(V) + \dim(V^\perp) = n$)
\( \nabla V \wedge V^\perp = 30 \). 

Question: How do you compute \( \text{Proj} \) (next time)?