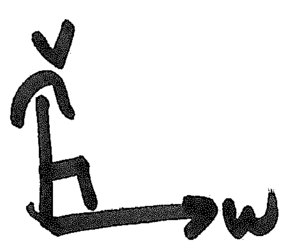
Lecture 15

Orthogonality + Projections in $\mathbb{R}^n$.

§1: Geometric Interpretations of dot products (revisited).

Thm: $v \cdot w = 0$ if and only if v and w are perpendicular.



Def: The length of a vector $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

is

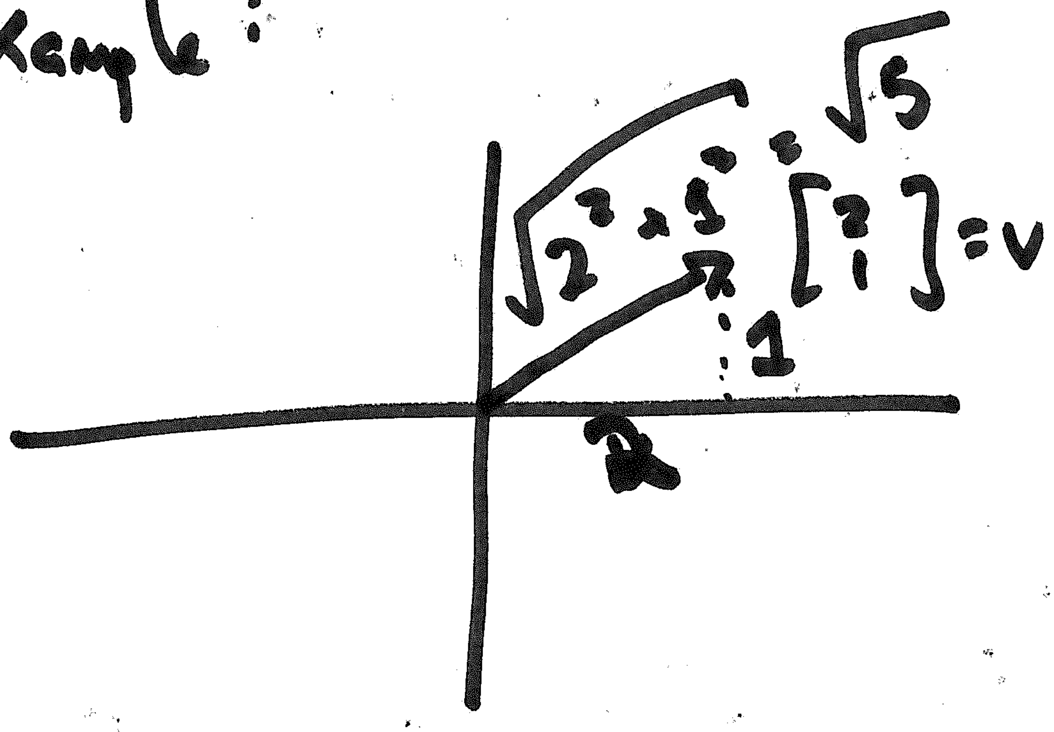
$$\|v\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$= \sqrt{v \cdot v}.$$

So $v \cdot v = \|v\|^2$. $\longrightarrow v$

$v \cdot v = 0 \iff \|v\| = 0 \iff \vec{v} = \vec{0}.$

Example:



$\|v\| = \sqrt{5}$.

Remark: Our <u>definition</u> of length <u>is</u> the Pythagorean theorem.

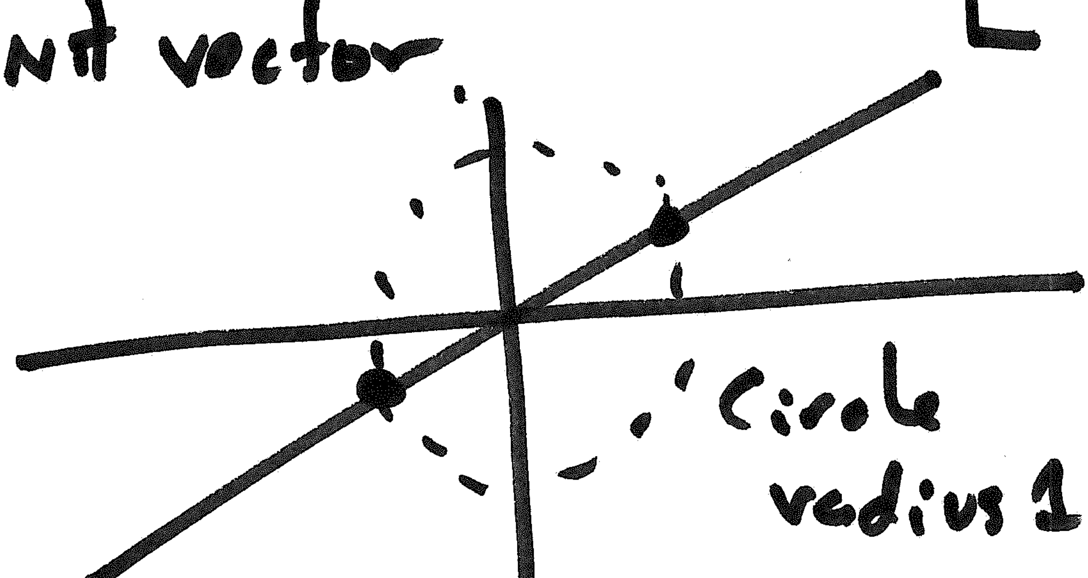
Def: We call $v \in \mathbb{R}^n$ a <u>unit vector</u> if

$\|v\| = 1$.

Thm: If v is non-zero

$v / \|v\|$ is

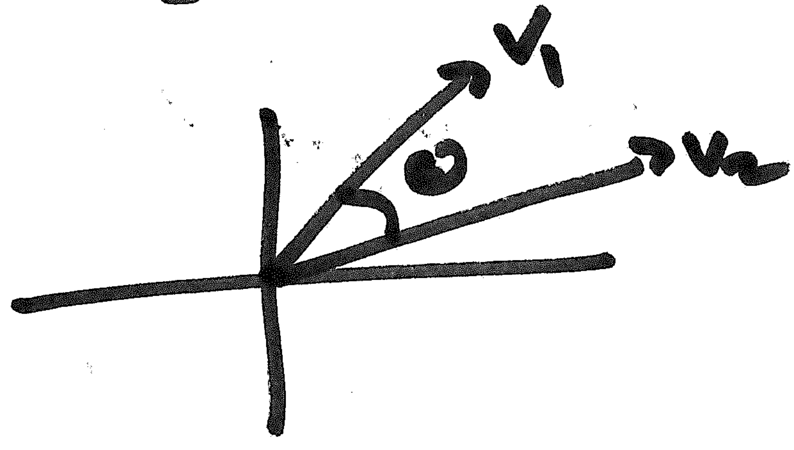
unit vector



Thm: Let $v_1, v_2 \in \mathbb{R}^n$
then

$$v_1 \cdot v_2 = \|v_1\| \, \|v_2\| \cos\theta$$

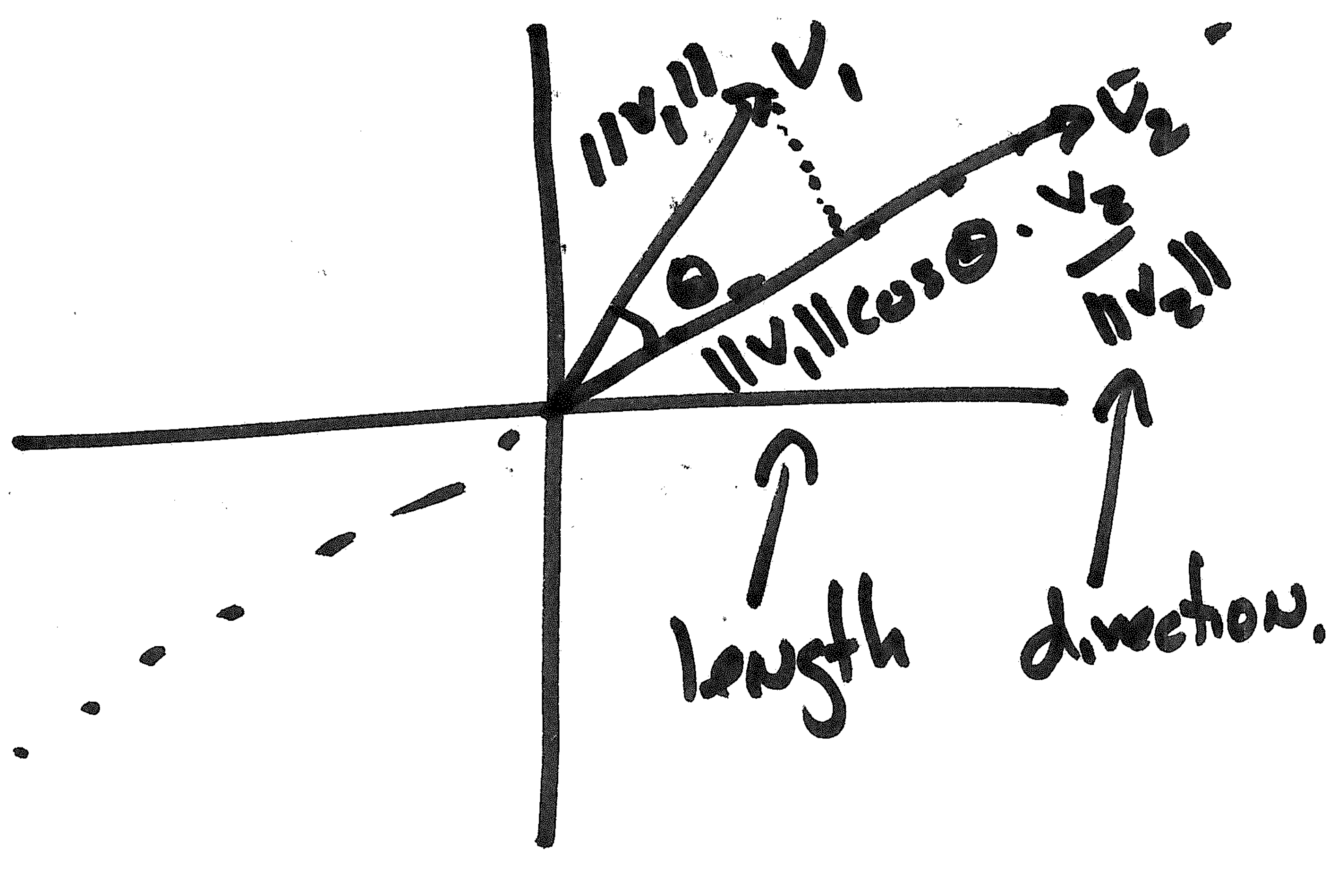
where θ is the angle between v_1 and v_2



where $0 \leq \theta < \pi$

Why is this formula true?

It relates two ways of describing projection onto a line.



So:

$$v_1^{\parallel} = \left(\frac{\|v_1\|}{\|v_2\|}\cos\theta\right) v_2$$

On the other hand, you know

$$v_1^{\parallel} = \left(\frac{v_1 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

Equating these:

$$\frac{v_1 \cdot v_2}{\|v_2\|^2} = \frac{\|v_1\|}{\|v_2\|}\cos\theta$$

So

$$v_1 \cdot v_2 = \|v_1\| \, \|v_2\| \cos\theta.$$

Problem: What is the angle between

$$v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ?$$

Using formula solve for θ.

$$v_1 \cdot v_2 = 1/2$$

$$\|v_1\| = \sqrt{3/4} = \frac{\sqrt{3}}{2}$$

$$\|v_2\| = 1$$

$$\frac{1}{2} = \frac{\sqrt{3}}{2} \cos\theta$$

$$\theta = \text{Arccos}\left(\frac{1}{\sqrt{3}}\right).$$

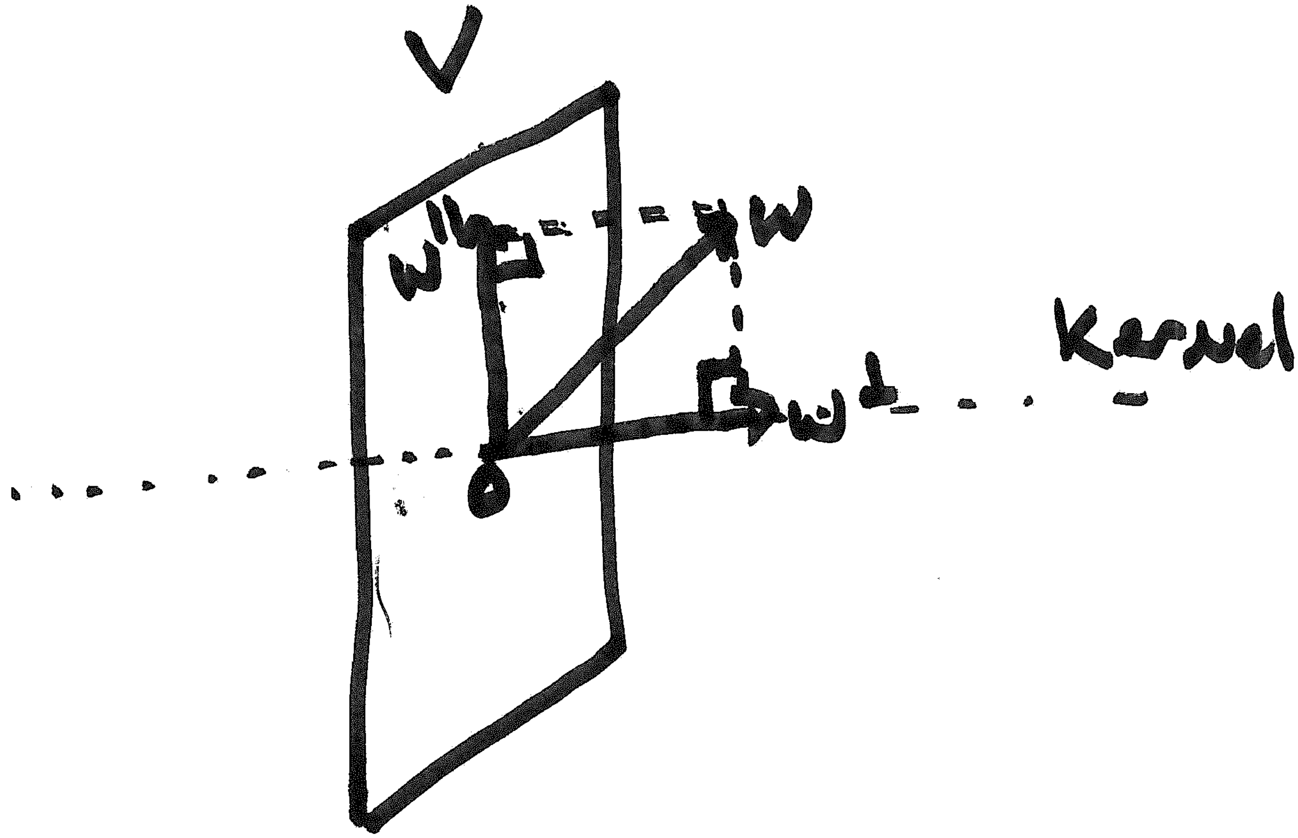
$\sim$ 54.7 degrees.

§2. Projection onto a subspace ~~&~~ + orthogonal complements.

Thm: Let $V \subseteq \mathbb{R}^n$ be a subspace, every vector $w \in \mathbb{R}^n$ can be written as

$$w = w^{\parallel} + w^{\perp}$$

where $w^{\parallel} \in V$ and $w^{\perp} \cdot \vec{v} = 0$ for all $\vec{v} \in V$ ($w^{\perp}$ is perp. to all $\vec{v} \in V$).



Thm / Def: The map

$$\mathrm{Proj}_V : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

defined by $w \longmapsto w^{\parallel}$

is called <u>orthogonal projection</u> <u>onto V</u>. Proj_V is a linear transformation.

Properties:

1. $\text{Proj}_V(\vec{v}) = \vec{v}$ if $\vec{v} \in V$.

2. $\text{IM}(\text{Proj}_V) = V$.

3. $w \in \ker(\text{Proj}_V)$ if $w^{\parallel} = 0$ so if $w = w^{\perp}$, i.e. w is perpendicular to all $\vec{v} \in V$.

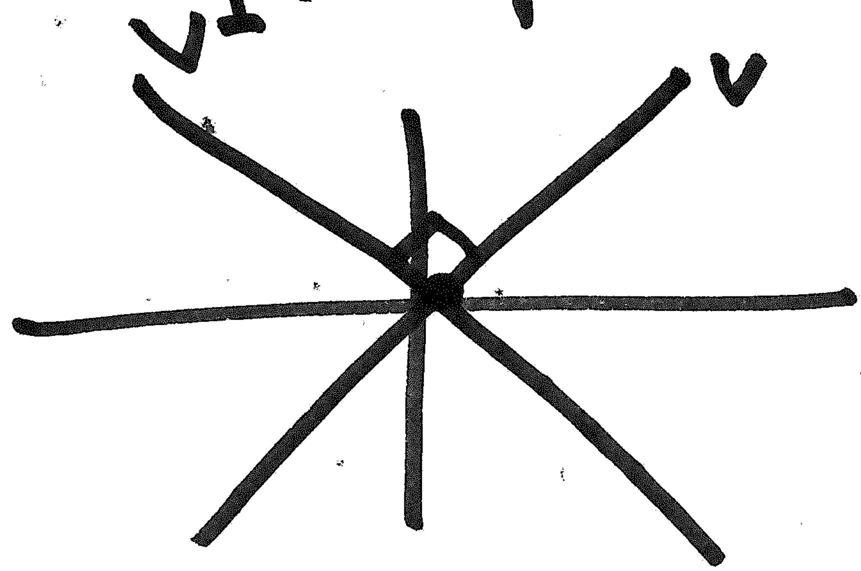
Def: The orthogonal complement of a subspace V is

$$V^{\perp} := \{ w \in \mathbb{R}^n : w \cdot \vec{v} = 0 \text{ for } \underline{\text{all}}\ \vec{v} \in V \}.$$

($V^{\perp}$ is the set of vectors perpendicular to <u>all</u> vectors in V.)

Examples of $V^\perp$

① In $\mathbb{R}^2$, if V is a line ~~then V~~ through 0 of slope m, then $V^\perp$ is perpendicular line (slope $-1/m$).



(2) In $\mathbb{R}^3$, if V is a plane through 0 then $V^\perp$ is the perpendicular line and vice versa.

(3) $(\mathbb{R}^n)^\perp = \{0\}$.

$\{0\}^\perp = \mathbb{R}^n$.

Properties of $V^\perp$

① $V^\perp$ is a subspace

(it's the kernel of the linear trans. Proj_V)

② $(V^\perp)^\perp = V$

③ $\dim(V^\perp) = n - \dim(V)$

(Rank nullity:

$$\underbrace{\dim(V)}_{\text{Image}} + \underbrace{\dim(V^\perp)}_{\text{kernel of } \text{Proj}_V} = n$$

(4) $V \wedge V^{\perp} = \{0\}$.

Question: How do you compute Proj (next time)?