Lecture 14

Similar Matrices.

Recall, let B be an $n \times n$ matrix and $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis for $\mathbb{R}^n$. Then the matrix

$$S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

transforms

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \text{ to } x$$

i.e. $x = S[x]_{\mathcal{B}}$

S^{-1} transforms x to β-coordinates

$$S^{-1}x = [x]_\beta .$$

In β-coordinates B is given by a matrix $[B]_\beta$.

i.e.

$$[B]_\beta [x]_\beta = Bx$$

and

$$S[B]_\beta S^{-1} = B.$$

Def: Let A and B be square $n \times n$ matrices. A and B are called <u>Similar</u> if there is an invertible matrix S such that

$$SAS^{-1} = B.$$

Example: $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

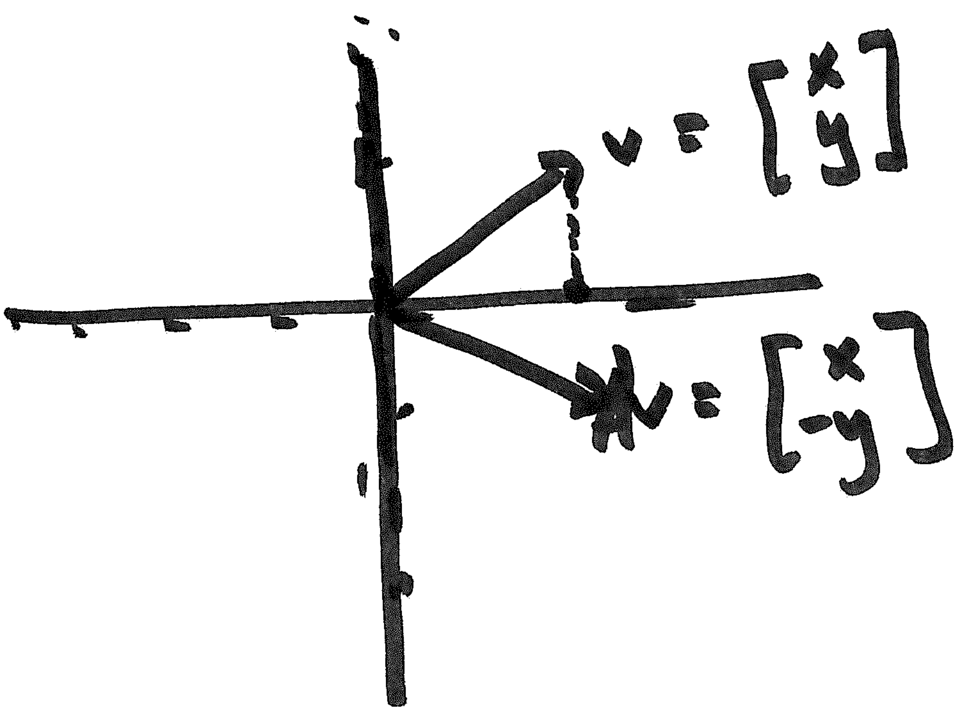
is similar to $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Why? Before we answer this let's think about A and B geometrically.

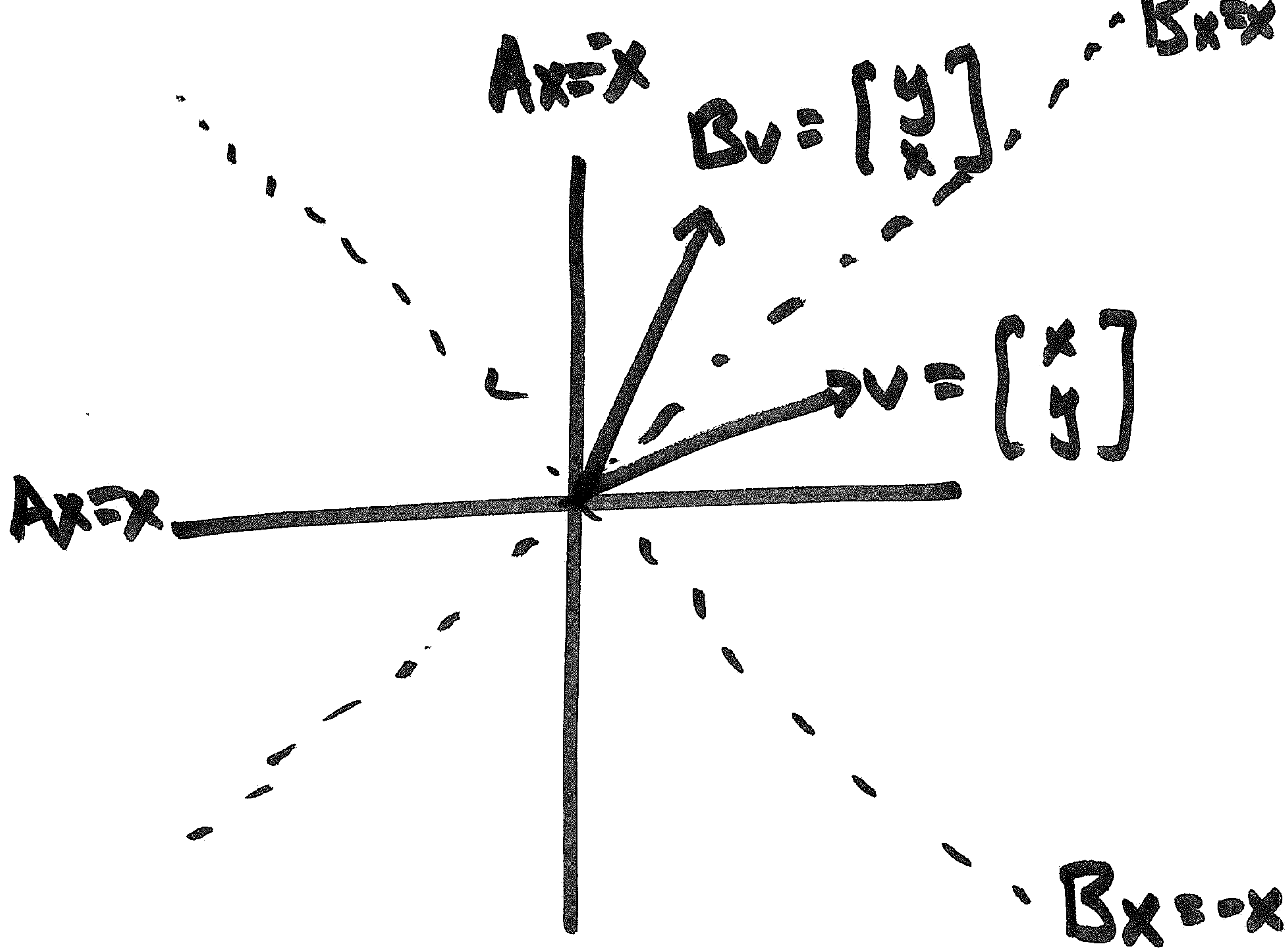
$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

Compute

$$Av = \begin{bmatrix} x \\ -y \end{bmatrix} \qquad Bv = \begin{bmatrix} y \\ x \end{bmatrix}.$$



A is a reflection over the x-axis



B is a reflection over the line of slope 1.

If S is a rotation that takes x-axis to the line of reflection and y-axis to the perpendicular line, then

Rotation 45°

S^{-1} takes the line of reflection to x-axis and takes perpendicular line to the y-axis

Rotation $-45°$.

rotate 45° rotate −45°

then

$$B = S A S^{-1}.$$

reflecting over line of slope 1

reflect over x-axis

S moves important directions for reflection to the <u>standard</u> directions of x, y axis.

What is S actually,

$$e_1 \xmapsto{S} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$e_2 \xmapsto{S} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} Se_1 & Se_2 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{1-(-1)\cdot 1}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

(Warning: this is not actually the rotation but a rotation + scaling)

$$B = S A S^{-1}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

B is A in coordinate system given by S

More generally, any reflection in $\mathbb{R}^2$ is similar to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Goal: understand a general linear transformation by finding the coordinate system in which it is most easily described geometrically.

Example 2:

$$A = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 \end{matrix} \end{array}.$$

Find SAS^{-1}.

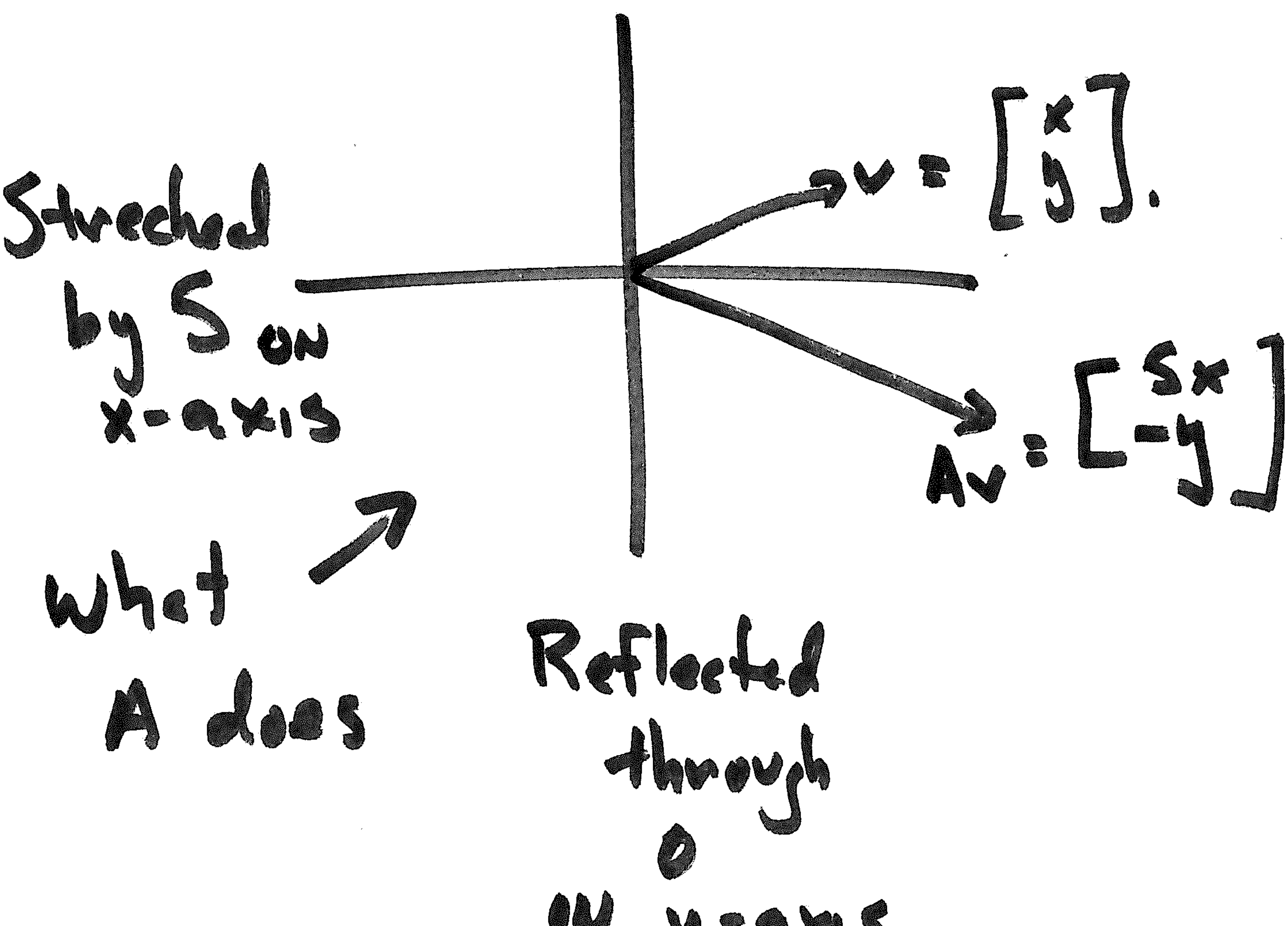
$$S^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \frac{1}{-3}.$$

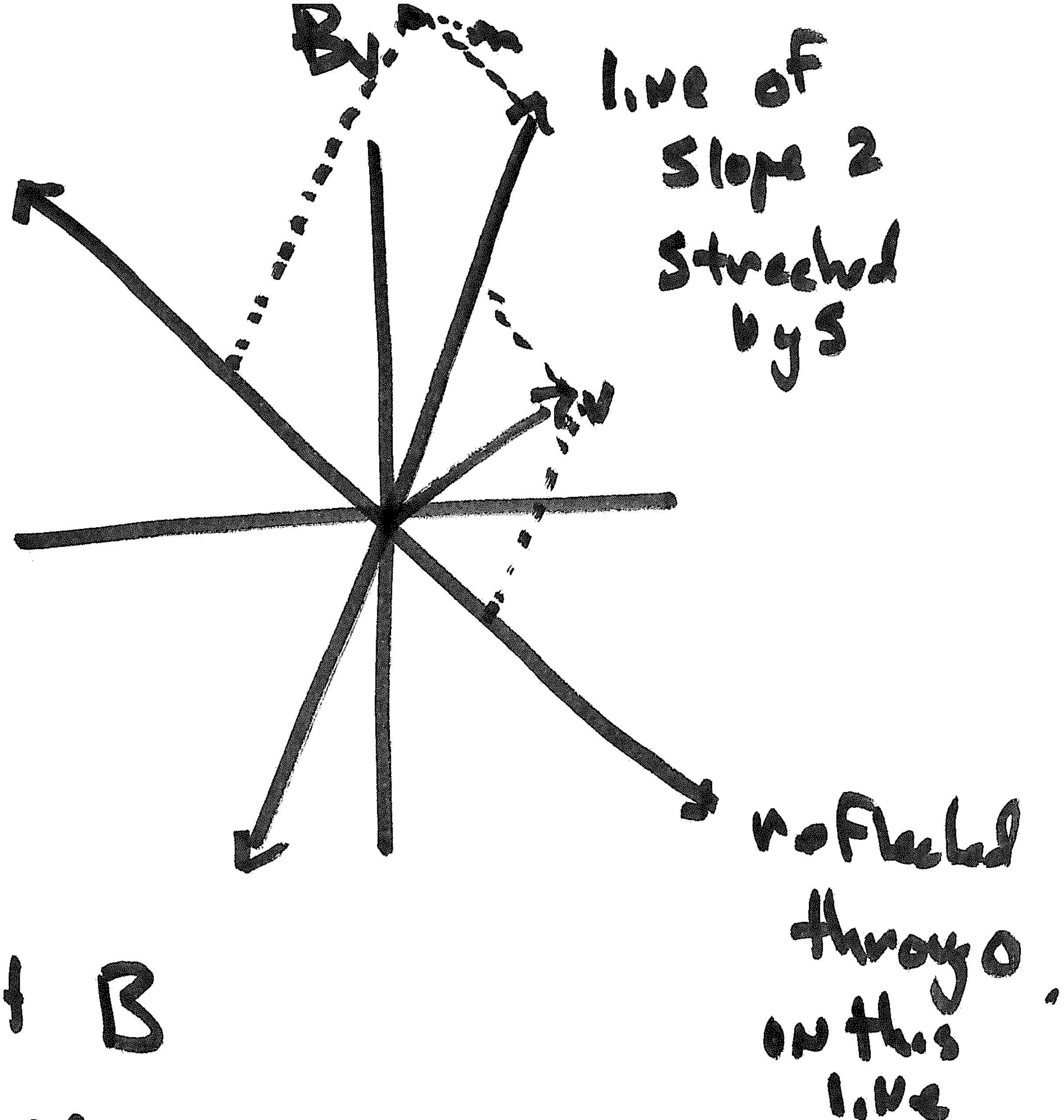
Compute

$$SAS^{-1} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = B.$$

A is similar to B.

We can understand B by understanding A.





What B does

Compute $B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\10\end{bmatrix} = 5\begin{bmatrix}1\\2\end{bmatrix}$

$B\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}+1\\-1\end{bmatrix} = -\begin{bmatrix}-1\\1\end{bmatrix}.$