Lecture 13

Coordinates.

Recall (Lecture 11)

Thm: Let $V \subseteq \mathbb{R}^n$ be a subspace and $v_1, \dots, v_n$ be a basis for V. Then every vector $\vec{x} \in V$ can written uniquely as

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

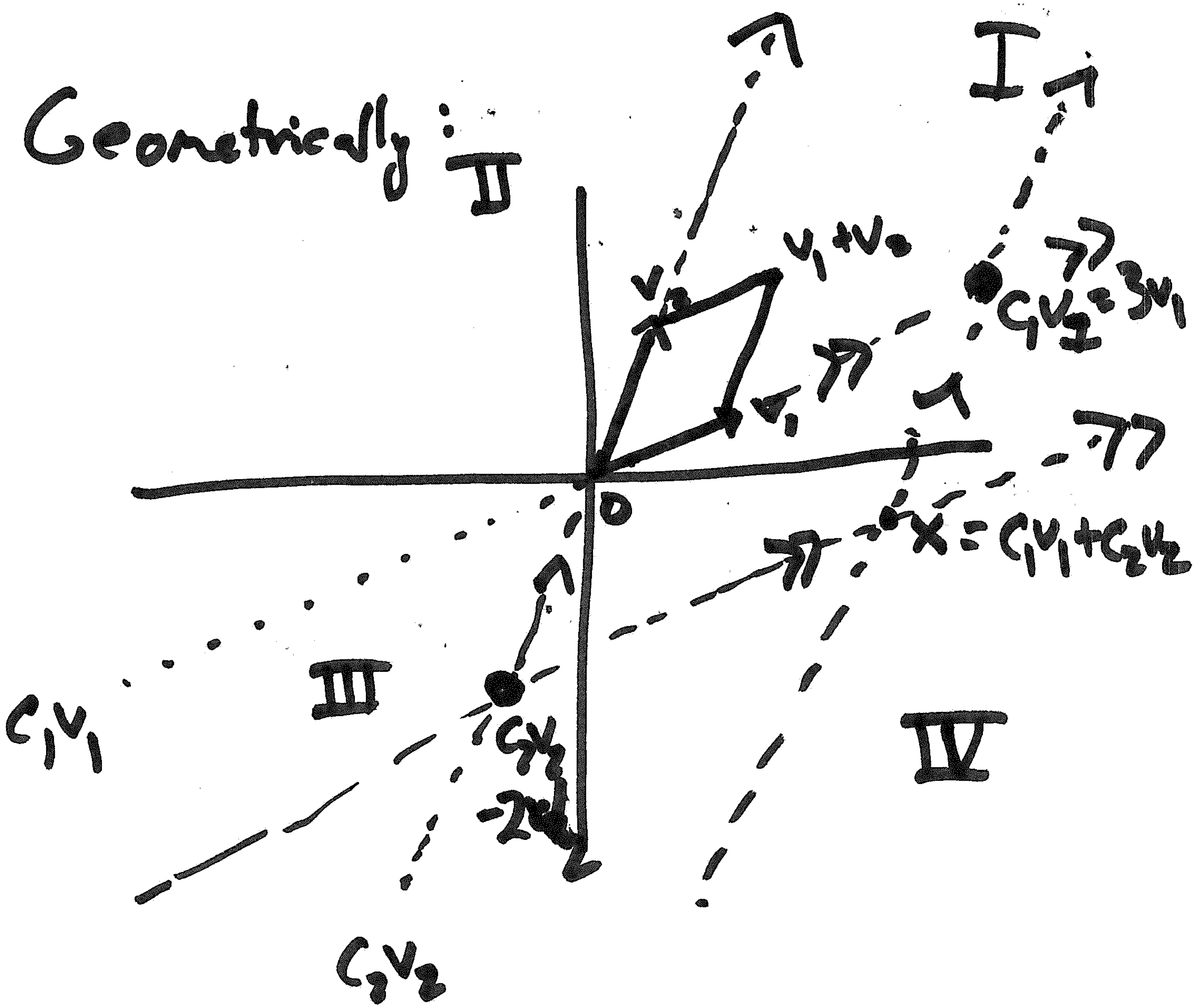
for $c_i \in \mathbb{R}$.

Example: $V = \mathbb{R}^2$

$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

Geometrically:



Algebraically:

Find $c_1, c_2 \in \mathbb{R}$ such that $c_1 v_1 + c_2 v_2 = x$.

$$\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} | & & | \\ c_1 v_1 & + & c_2 v_2 \\ | & & | \end{bmatrix} = x$$

$$x = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 2 + 1 c_2 \\ c_1 \cdot 1 + 2 \cdot c_2 \end{bmatrix}$$

Solving this matrix equation is the same as solving the system encoded by

$$\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 2 & -1 \end{array}\right].$$

Row reduce $\Rightarrow$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right].$$

$c_1 = 3$ and $c_2 = -2$.

So to describe the location of x given a basis

$$\beta = (v_1, \ldots, v_n)$$

the information I need is a list of numbers

$$c_1, \ldots, c_n$$

Such that

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

I record this list as a vector.

Def: Let $V \subseteq \mathbb{R}^n$ be a subspace of and $\beta = (v_1, \ldots, v_n)$ be a basis for V. Let $x \in V$.

The <u>β-coordinate vector of $x \in V$</u> is the unique vector

$$[x]_\beta := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

such that

$$c_1 v_1 + \ldots + c_n v_n = x.$$

Example: $\beta = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. $\leftarrow$ Standard basis

Then

$[x]_\beta = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. $\leftarrow$ basis β.

Example: $\beta = (e_1, \dots, e_n)$ the standard basis for $\mathbb{R}^n$ and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ then

$$[x]_\beta = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Given a vector x in β coordinates how do I recover the vector in standard coordinates and vice versa?

this takes $[x]_\beta$ to x in standard.

Observe:

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

where $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [x]_\beta$

I can express this equation in terms of a matrix product.

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \begin{bmatrix} c_1 v_1 + \cdots + c_n v_n \\ \\ \end{bmatrix}$$

$$= x.$$

That is to transform from β-coordinates to standard coordinates you multiply by $S = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$.

$$x = S\,[x]_\beta .$$

Example: $\beta = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$x = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad [x]_\beta = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

What is S?

$$S = \begin{array}{c} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \quad v_1 \quad v_2 \end{array}$$

Observe that

$$S[x]_\beta = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix} = x \checkmark$$

To go from standard coordinates to $\mathcal{B}$ coordinates multiply by S^{-1}.

$$S^{-1}x = [x]_{\mathcal{B}}.$$

Example: $S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} =$

$$\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}.$$

$$S^{-1}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Every basis of $\mathbb{R}^N$ gives a coordinate system on $\mathbb{R}^N$.

Why should you care about the non-standard coordinate systems?

Choosing the right coordinate system may make complicated things simpler

In this class, we'll be interested in finding the right coordinate system to simplify a linear transformation.

We've already done this in one case.

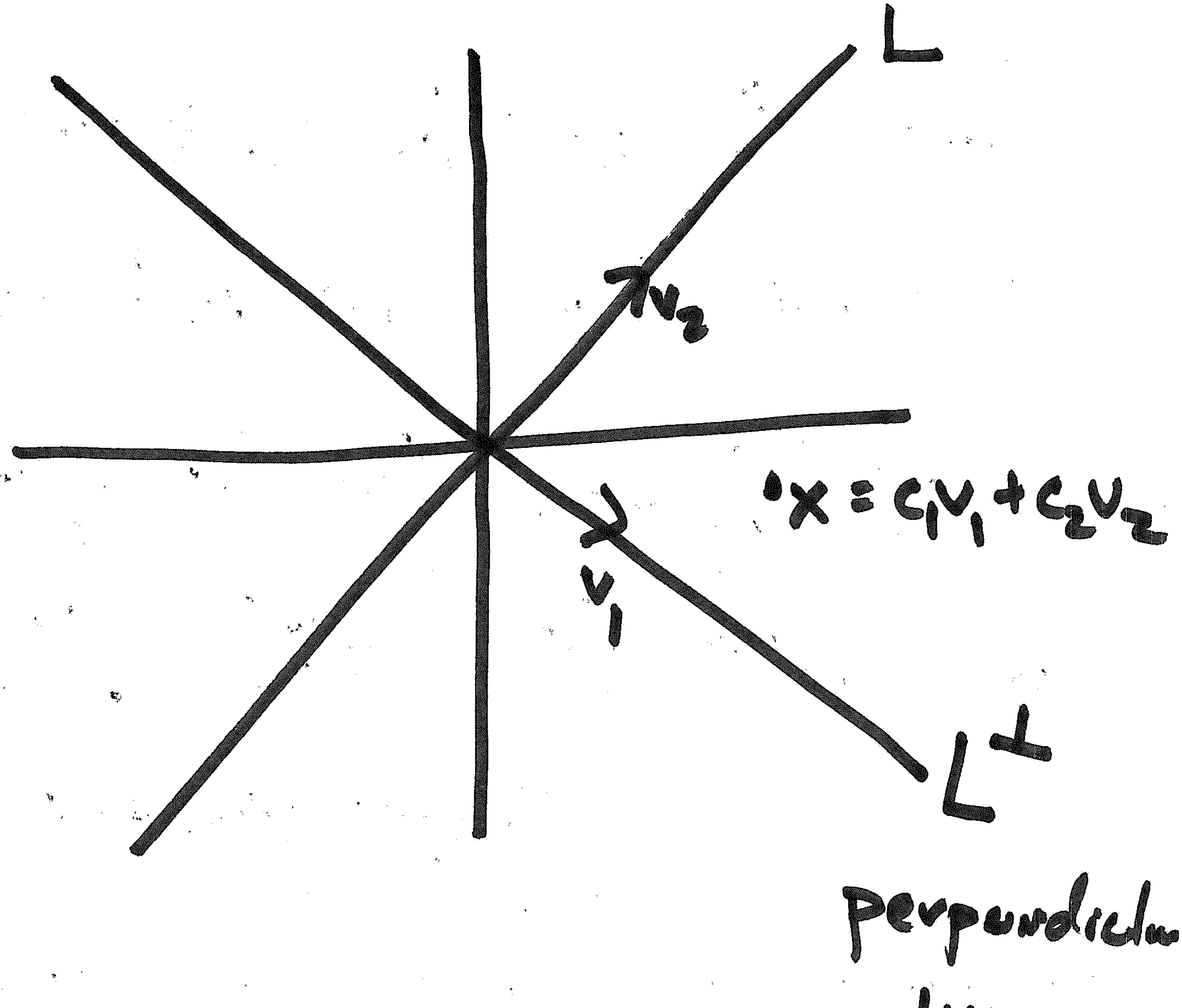
Reflection over a line L

in standard coordinates

$$\frac{1}{x^2+y^2}\begin{bmatrix} x^2-y^2 & 2xy \\ 2xy & -y^2+x^2 \end{bmatrix}$$

where $\begin{bmatrix} x \\ y \end{bmatrix} \in L$.

But there is a way of simplifying this by considering the right coordinate system.



In terms of (v_1, v_2)-coordinates reflection over L is given by

$$Rx = -c_1 v_1 + c_2 v_2.$$

$\mathcal{B} = (v_1, v_2)$

$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$R([x]_{\mathcal{B}}) = \begin{bmatrix} -c_1 \\ c_2 \end{bmatrix} = [Rx]_{\mathcal{B}}.$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

IN $\mathcal{B}$-coordinates reflection over L is given by multiplication by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Def: Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a linear transformation and β be a basis for $\mathbb{R}^N$.

The β-matrix for T is the unique matrix $[T]_\beta$ such that

$$[Tx]_\beta = [T]_\beta [x]_\beta .$$

i.e. the matrix $[T]_\beta$ describes the transformation T in β-coordinates.

Example:

In $\beta = \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} -y \\ x \end{bmatrix} \right)$

coordinates the reflection

$$T = \frac{1}{x^2+y^2} \begin{bmatrix} x^2-y^2 & 2xy \\ 2xy & y^2-x^2 \end{bmatrix}$$

satisfies

$$[T]_\beta = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}.$$

How do you find $[T]_\beta$?

$$\begin{bmatrix} Te_1 & Te_2 & \cdots & Te_n \end{bmatrix}$$

is matrix of T in standard basis.

In β-coordinates

$$[T]_\beta = \begin{bmatrix} [Tv_1]_\beta & [Tv_2]_\beta & \cdots & [Tv_n]_\beta \end{bmatrix}$$

Way 2: T given by matrix A in standard coordinates

translate back $\qquad\qquad$ translate

$$A = S\,[T]_\beta\,S^{-1}$$

$$[T]_\beta = S^{-1} A S.$$