Lecture 13

Rank-Nullity Theorem
Let $A$ be a matrix.

Procedure to find a basis for the kernel of $A$.

**Step 1**: Solve $Ax = 0$.

- Row reduce $A$
  $$A \rightarrow \text{rref}(A).$$

- Unpack back to equations. $\text{rref}(A)$ encodes equations which describe how a set of dependent variables depend on free variables.

The kernel of $A$ is the set of vectors obtained by every choice of values for the free variables.
Step 2: Solving $Ax = 0$ gives the kernel as a set of vectors whose entries are linear expressions of the free variables. Let's call these variables $t_1, \ldots, t_d$ (e.g., $t_1 = x_2$, $t_2 = x_4$). We may factor these expressions as

$$t_1v_1 + t_2v_2 + \ldots + t_dv_d$$

where $v_i$ is a vector in $\mathbb{R}^n$ consisting of coefficients.

The vectors $v_1, \ldots, v_d$ are a basis for $\ker(A)$. 
Example:

\[ A = \begin{bmatrix}
-1 & 2 & -1 & -5 & c \\
2 & -2 & 9 & 5 & -8 \\
3 & 6 & 1 & 5 & 7
\end{bmatrix} \]

Step 1:

\[ A \Rightarrow rulcF(A) = \begin{bmatrix}
1 & 2 & 0 & 3 & 0 \\
0 & 0 & 1 & -4 & 0 \\
0 & 0 & 0 & a & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

Unpack:

\[ x_1 + 2x_2 + 3x_4 = 0 \iff x_1 = -2x_2 - 3x_4 \]
\[ x_3 + 4x_4 = 0 \iff x_3 = 4x_4 \]
\[ x_5 = 0 \iff x_5 = 0 \]

\[ x_1, x_3, x_5 \text{ dependent} \]
\[ x_2, x_4 \text{ free.} \]
\[ \text{Ker}(A) = \left\{ \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ 4x_4 \\ x_4 \\ 0 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\} \]

**Step 2.**

\[
\begin{bmatrix}
-2x_2 - 3x_4 \\
x_2 \\
4x_4 \\
x_4 \\
0
\end{bmatrix} = x_2 \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
0 \\
0 \\
4 \\
1 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-3 \\
0 \\
4 \\
1 \\
0
\end{bmatrix}
\]

is a basis for \text{Ker}(A).
Why does this procedure yield a basis?

We need to check two things.

**Span**: the vectors \( v_1, \ldots, v_d \) span because every element of kernel was written in the form \( t_1v_1 + \ldots + t_dv_d \). (Get span for free)

**Linearly**: If \( t_1v_1 + \ldots + t_dv_d = 0 \) then recombine left hand side and see that each of the free variables appear as an entry. So \( t_i = 0 \).
Linear Independence in our first example

What does it mean for \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) to be linearly independent?

The definition says if

\[
\mathbf{0} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2
\]

then \( c_1, c_2 = 0 \).

i.e. the only way to write

\[
\mathbf{0} = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2
\]

is when \( x_1, x_2 = 0 \).

If

\[
\mathbf{0} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ 0 \\ 0 \end{bmatrix}
\]

so \( x_2 = 0 \), \( x_4 = 0 \).
What is the dimension of the kernel of $A$?

In our example, the dimension is 2 since $v_1, v_2$ are a basis.

In general, the dimension of the kernel is the number of free variables.

Last time we learned about the $\text{rank}(A) = \# \text{ pivots in ref} = \# \text{ of dependent variables.}$
Theorem (Rank-Nullity Theorem)

Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then

$$m = \dim(\mathbb{R}^n) = \text{Rank}(A) + \text{Nullity}(A)$$

$$= \dim(\text{Im}(A)) + \dim(\text{Ker}(A)),$$

where $\text{Nullity}(A) = \dim(\text{Ker}(A))$.
Last class

\[ \text{dim } (\text{Im} (A)) = \text{rank } (A) = \# \text{ pivots} \]

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

base \( C \subseteq \text{Im} (A) \).

\[ \text{rref } (A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]
How to think about rank-nullity theorem:

The rank-nullity theorem states

\[ m = \text{rank}(A) + \text{nullity}(A) \]

\[ \begin{array}{c}
\text{dimension of domain} \\
\text{dimension of image}
\end{array} \]

\[ \begin{array}{c}
\text{dimension of kernel} \\
\text{dimension of kernel}
\end{array} \]

A map \( A \) as:

\[ \text{Input} = \text{Output} + \frac{\text{Information}}{\text{Loss}} \]
Theorem: Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Let $b \in \text{Im}(A)$.

The solution set to

$$A x = b$$

is the translate of the kernel of $A$ by any solution $x_0$, i.e.,

$$\exists x' \mid Ax' = 0 \exists z = x' + x_0 \mid x' \in \ker(A) \quad Ax_0 = b.$$
Exemple: $A = \begin{bmatrix} 2 & 1 \end{bmatrix}$

$Ax = b \quad \iff \quad 2x + y = b$

$A: \mathbb{R}^2 \rightarrow \mathbb{R}$
Why is this true?
Fix \( x_0 \) such that \( Ax_0 = b \).
We want to show

\[
\sum \{ Ax = b \} = x_0 + \text{Ker}(A)
\]

Assume \( x_0 + x' \in x_0 + \text{Ker}(A) \).

\[
A(x_0 + x') = Ax_0 + Ax' \\
= b + 0 \\
= b
\]

So \( x_0 + x' \) is a solution to \( Ax = b \).

If \( x \) is a solution to \( Ax = b \),

\[
A(x - x_0) = b - b = 0
\]

so \( x - x_0 \in \text{Ker}(A) \) so \( x = x_0 + (x - x_0) = x_0 + \text{Ker}(A) \).
Invertible matrices  \( \mathbf{N} = \mathbf{M} \)

\[
\frac{\text{rank-nullity}}{\text{rank-nullity}}
\]

Let \( \mathbf{A} \) be a \( n \times n \) matrix.

Then: The following are equivalent:

(i) \( \mathbf{A} \) is invertible.

(ii) \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has a unique solution \( \mathbf{x} \) for all \( \mathbf{b} \in \mathbb{R}^n \)

\( \mathbf{x} = \mathbf{A}^{-1} \mathbf{b} \).

(iii) \( \ker(\mathbf{A}) = \{ \mathbf{0} \} \).

(iv) \( \text{Im}(\mathbf{A}) = \mathbb{R}^n \).

(v) \( \text{rank}(\mathbf{A}) = n \).

(vi) \( \text{nullity}(\mathbf{A}) = 0 \) \quad \text{equivalent by rank-nullity thm.}

(vii) \( \text{Rref}(\mathbf{A}) = \mathbf{I} \).
(viii) The columns of $A$ are L.I. ($\ker = \{0\}$)

(ix) The columns of $A$ are a basis

(x) The columns of $A$ span $\mathbb{R}^n$.

(Image is $\mathbb{R}^n$)