Lecture 12

The rank-nullity theorem.
General Problem: Given a matrix $A$ find bases for $\ker(A)$ and $\text{Im}(A)$. 
Thm: The columns of $A$ which contain a pivot upon row reduction form a basis for the image of $A$. 
Example (from lecture 10)

$$A = \begin{bmatrix} 1 & 2 & 2 & 6 \\ -1 & -2 & -1 & 1 \\ 4 & 8 & 6 & 5 \\ 3 & 2 & -8 & 7 \end{bmatrix}$$

$$R_{\text{ref}}(A) = \begin{bmatrix} \star & \star & \star & \star \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\text{In}(A) = \text{Span} \left( \begin{bmatrix} -1 \\ -4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 9 \\ 7 \end{bmatrix} \right)$

Basis and these columns are linearly independent.
Warning: The image of $A$ is not spanned by the columns of $r\text{ref}(A)$ (in general).

Ex: $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\text{Span}\{[1, 2]\} = \text{Im}(A)$ is a line through $0$ of slope $2$.

$r\text{ref}(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not on this line.
Why is this theorem true?

To be a basis, we need to check vectors are L.I. and span the image.

Linear independence

Last time we saw $v_1, \ldots, v_n$ are L.I. if and only if

$\text{ref}([v_1, \ldots, v_n])$ has every column containing a pivot.
Starting with any matrix $A$ and taking only those columns $v_1, \ldots, v_n$ that contain a pivot upon row reduction implies

$\text{ref}\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$

has all columns containing a pivot. So $v_1, \ldots, v_n$ are L.I.
Why do these columns span?

We know that all columns span $\text{Im}(A)$. But we don't need a column $v_n$ if $v_n$ is in $\text{Span}(v_1, \ldots, v_{n-1})$.

Def: A vector $v_n$ is called redundant on $v_1, \ldots, v_{n-1}$ if $v_n \in \text{Span}(v_1, \ldots, v_{n-1})$.

i.e. we don't need redundant columns in our spanning set.
Claim: If a column does not contain a pivot its redundant.

Why? We have to find $c_1, \ldots, c_{n-1}$ such that

\[ c_1 v_1 + c_2 v_2 + \ldots + c_{n-1} v_{n-1} = v_n. \]

This is a system of linear equations corresponding to the augmented matrix containing the first $n$ columns. Not a pivot $\Rightarrow$ system is consistent.
E.g. \( v_2 \) is not a pivot

\[
\begin{bmatrix}
1 \\
v_2
\end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
\]

\[2v_1 = v_2\]

\( v_4 \) is not a pivot

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\( c_1 + 2c_2 = 3, \ c_3 = -4 \)
Since I only need one solution consider $c_2 = 0$, we see

$$3v_1 - 4v_3 = v_4.$$  

Columns not containing pivot are redundant $\Rightarrow$ you can spin the image without them.
What is the dimension of $\text{Im}(A)$?

It's the number of vectors in any basis.

Def: The rank of an $n \times m$ matrix is the number of pivots in $\text{rref}(A)$. It's denoted by $\text{rank}(A)$. 
Thm: $\dim(\text{Im}(A)) = \text{Rank}(A)$.

E.g. in example

$\dim(\text{Im}(A)) = 3.$

$\text{Im}(A) \subseteq \mathbb{R}^4$

What is $k$? $k = 4$

$\text{Im}(A)$ is smaller than $\mathbb{R}^4$ (in dimension) so $\text{Im}(A) \neq \mathbb{R}^4$.

So there is a $b \in \mathbb{R}^4$ such that $Ax = b$ does not have a solution.
Consequence: If $A$ is an \( n \times m \) matrix, i.e., a linear transformation \( A: \mathbb{R}^m \rightarrow \mathbb{R}^n \), then \( \text{Im}(A) \subseteq \mathbb{R}^n \) so \( \dim(\text{Im}(A)) = \text{Rank}(A) \leq n \), and \( \text{Rank}(A) = n \) if and only if \( \text{Im}(A) = \mathbb{R}^n \) i.e. $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$.

Ex: $A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. 

\[ \text{Im}(A) = \{0\} \] 
\[ \dim(\text{Im}(A)) = 0 < n. \]
What about the kernel of $A$?

We already secretly know how to do this.

Finding the kernel is solving $Ax = 0$.

This is solving a system of linear equations.

Step 1: Row reduce $[A|0]$ such nothing happens in $0$ column. Row reduce $A$. We find $A$ and $\text{rref}(A)$ have same solution set i.e. $\text{Ker}(A) = \text{Ker}(\text{rref}(A))$. 
Step 2: \( \text{rref}(A) \) tells us how to solve for "dependent" variables in terms of free variables.

In our example:

\( x_1, x_3, x_5 \) dependent

\( x_4, x_5 \) free.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{bmatrix}
= \begin{bmatrix}
  -2x_2 & -3x_4 \\
  x_2 & 4x_4 \\
  0 & x_4
\end{bmatrix}
\]

is with kernel if

\[
\begin{align*}
  x_1 + 2x_2 + 3x_4 &= 0 \\
  x_3 + (-4)x_4 &= 0 \\
  x_5 &= 0
\end{align*}
\]
We can rewrite this as:

\[ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \end{bmatrix}. \]

\(\textcircled{3}\) The vectors multiplied by the "free" variables span the kernel.

But they are also linearly independent. Why?
$c_1 \left[ \begin{array}{c} -2 \\ 1 \\ 0 \end{array} \right] + c_2 \left[ \begin{array}{c} -3 \\ 0 \\ 4 \\ 0 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$

$c_1 = c_1 \cdot 1 + c_2 \cdot 0 = 0$
$c_2 = c_1 \cdot 0 + c_2 \cdot 1 = 0$

$\Rightarrow$ vectors are L.I.

Claim: This process (which you know) always yields a basis.
What is the dimension of \( \ker(A) \)?

It's the number of "free" variables.

It's the total # of variables — # pivots.

Thm: \( \dim(\ker(A)) = M - \text{rank}(A) \),

where \( A: \mathbb{R}^m \to \mathbb{R}^n \).
Thm (Rank - Nullity Theorem)

Let $A$ be an $n \times m$ matrix (i.e., a linear transformation $A: \mathbb{R}^m \to \mathbb{R}^n$)

then

$\dim(\ker(A)) + \dim(\text{Im}(A)) = \dim(\mathbb{R}^m) = \dim(\mathbb{R}^n)$

Equivalently,

$\text{Rank}(A) + \text{Nullity}(A) = n$

Where $\text{Nullity}(A) = \dim(\ker(A))$. 
Thm: \( \dim(\ker(A)) = 0 \) if and only if \( Ax = b \) has at most 1 solution for all \( b \in \mathbb{R}^n \).

Why? If \( x_1, x_2 \) are solutions to \( Ax = b \). Then

\[
A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0
\]

So \( x_1 - x_2 \in \ker(A) \) which if \( \ker(A) = \{0\} \) implies

\[
x_1 - x_2 = 0 \implies x_1 = x_2.
\]
Theorem: Let $A$ be an $n \times n$ matrix. The following are equivalent.

1. $A$ is invertible
2. $\text{Ref}(A) = I$.
3. For all $b \in \mathbb{R}^n$ there is a unique solution $x \in \mathbb{R}^n$ to $Ax = b$.
4. $\ker(A) = \{0\}$ or $\dim(\ker(A)) = 0$.
5. $\text{Im}(A) = \mathbb{R}^n$ or $\dim(\text{Im}(A)) = n$.
6. $\text{Rank}(A) = n$.
7. The columns of $A$ span $\mathbb{R}^n$ (or a basis, or linearly independent).