Lecture 11

Bases and Dimension.

Motivating Thm

Let v_1 and v_2 be a pair of vectors in $\mathbb{R}^2$ such that

$\text{Span}(v_1)$ and $\text{Span}(v_2)$

are distinct lines (through 0).

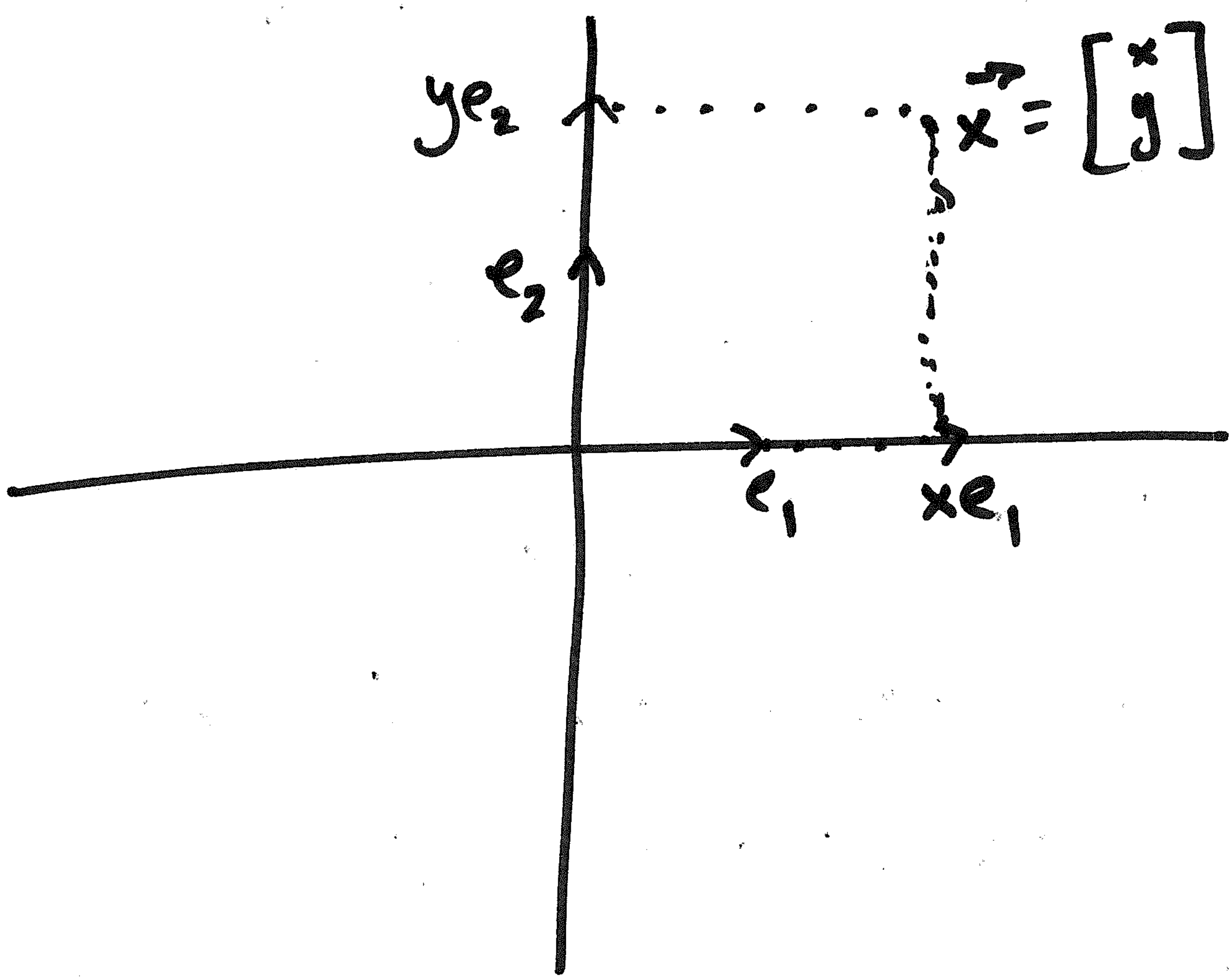
Then every vector $\vec{x} \in \mathbb{R}^2$ can be expressed as

$$c_1 v_1 + c_2 v_2 = \vec{x}$$

for some __unique__ constants c_1 and c_2.

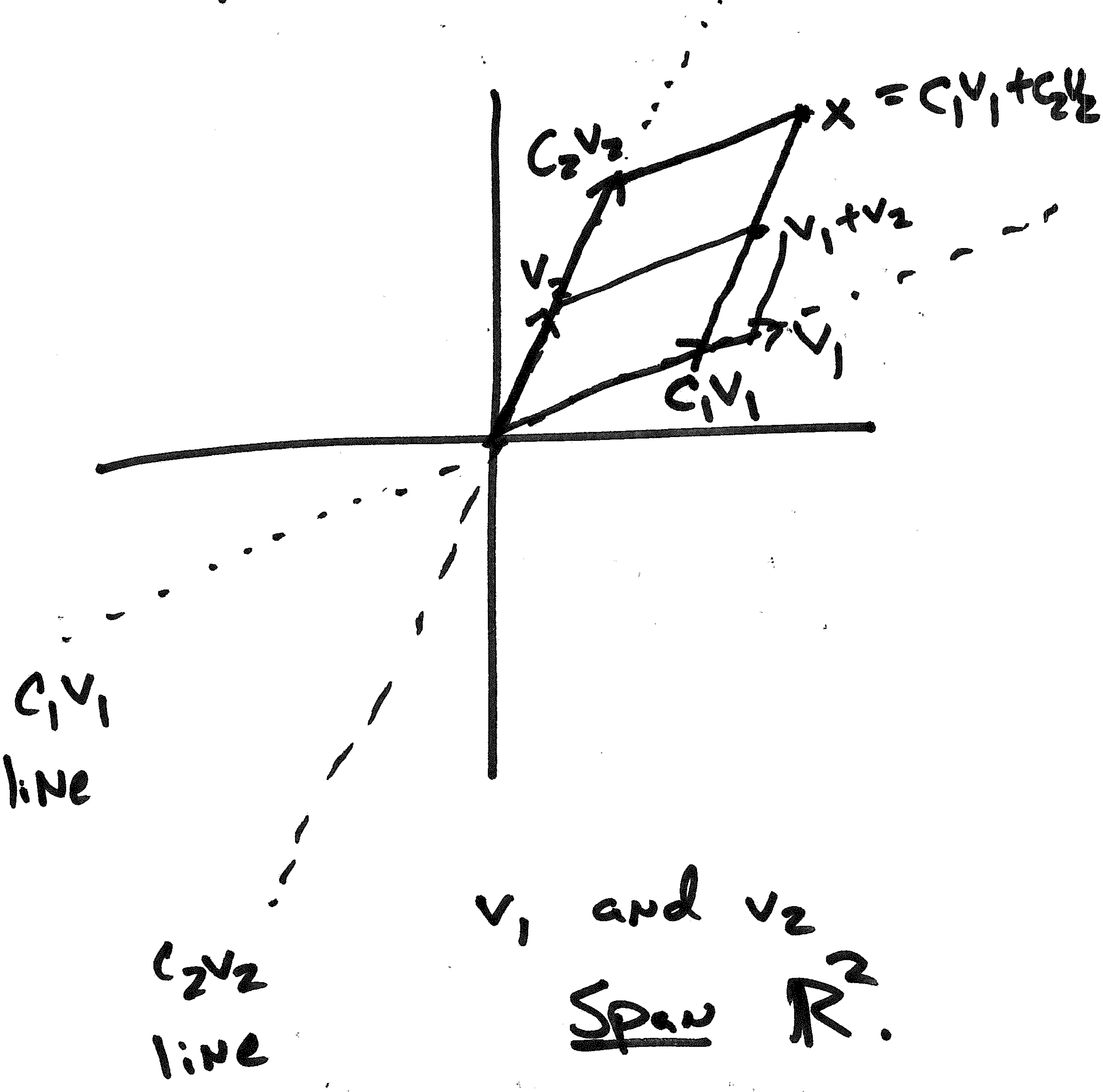
Exemple: $v_1 = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$v_2 = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$\vec{x} = x e_1 + y e_2.$$

Example: More generally



v_1 and v_2 <u>Span</u> $\mathbb{R}^2$.

We may think of c_1 and c_2 as the coordinates of $\vec{x}$ in a coordinate system <u>based</u> on the location of v_1 and v_2.

Today, we'll describe for a general subspace $V \subseteq \mathbb{R}^k$ sets of vectors on which a coordinate system may be <u>based</u>.

These are called <u>bases</u>.

Def: A set of vectors

$$v_1, \dots, v_d \in \mathbb{R}^k$$

are said to be linearly indendent if the only expession for the zero vector

$$\vec{0} = c_1 v_1 + c_2 v_2 + \dots + c_d v_d$$

is when all scalars $c_i = 0$.

Ex: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots,$

$e_K = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

are linearly indendent.

Why?

$$c_1 e_1 + c_2 e_2 + \ldots + c_K e_K = \vec{0}$$

$$\| $$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{bmatrix} \Longrightarrow c_i = 0$$

fr all i.

Ex: $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}$

are linearly indendent.

Why? If we want to find c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = 0.$$

That's the same as finding solutions

$$\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0}.$$

To find solutions we row reduce

$$A = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 2 & 1 \\ 3 & 5 \end{bmatrix}$$

and find

$$\text{Rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 = 0 \qquad c_2 = 0.$$

Remark: $v_1, \ldots, v_k$ are linearly independent if and only if

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

has kernel $\operatorname{Ker}(A) = \{\vec{0}\}$

How to think about Liner Independence:

If $v_1, \ldots, v_K$ are L.I.

then
$$\vec{0} \neq -v_1 + c_2 v_2 + \ldots + c_K v_K.$$
for any $c_2, \ldots, c_K$.

Equiv.
$$v_1 \neq c_2 v_2 + \ldots + c_K v_K$$
i.e.
$$v_1 \notin \mathrm{Span}(v_2, \ldots, v_K).$$

Thm: More generally $v_1, \ldots, v_k$ are linearly independent $\Longleftrightarrow$ v_i is not in the span of the remaining v_j for all $i \leq k$.

How do you determine if $v_1, \ldots, v_k$ are linearly independent?

Thm: $v_1, \ldots, v_k$ are linearly independent if

$$\mathrm{rref}\left(\begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}\right) = \text{``}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\text{''}$$

i.e. every column of rref of the matrix with $v_1, \ldots, v_k$ as columns contains a <u>pivot</u>.

Def: A **basis** **for a subspace** $V \subseteq \mathbb{R}^K$ is an **ordered** collection of linearly independent vectors

$$v_1, \ldots, v_K$$

such that

$$\mathrm{Span}(v_1, \ldots, v_K) = V.$$

Examples:

1. $e_1, \dots, e_k$ in $\mathbb{R}^k$ this is a <u>basis</u> <u>for</u> <u>$\mathbb{R}^k$</u>.

2. e_2, e_1 is a basis for $\mathbb{R}^2$.

   Different than e_1, e_2

   Order matters.

3. If v_1 and v_2 are not colinear than v_1 and v_2 are a basis for $\mathbb{R}^2$.

Non-examples:

1. $\begin{bmatrix}1\\2\end{bmatrix} \in \mathbb{R}^2$ not a basis for $\mathbb{R}^2$.

   $\operatorname{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ is a line so not $\mathbb{R}^2$.

   $\begin{bmatrix}1\\2\end{bmatrix}$ is linearly indep't.

   $$c\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

   $$\Longleftrightarrow c = 0.$$

2. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$ span $\mathbb{R}^2$ but are not L.I.

Ex: 4

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}7\\1\\5\end{bmatrix}$$

is a basis for the plane

$$\operatorname{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}7\\1\\5\end{bmatrix}\right) \subseteq \mathbb{R}^3.$$

Thm: Let $V \subseteq \mathbb{R}^k$ be a subspace and $v_1, \ldots, v_d$ be a basis for V. Then every vector $x \in V$ can be expressed uniquely as

$$x = c_1 v_1 + \ldots + c_d v_d$$

with $c_i \in \mathbb{R}$.

why? Given $x \in V$ we want to find $\begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$ such that $c_1 v_1 + \dots + c_d v_d = x$.

We want to solve the system given

$$A = \left[\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_d & x \\ | & | & & | & | \end{array}\right]$$

Row reduce:

$$\operatorname{rref}(A) = \left[\begin{array}{cccc|c} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ 0 & 0 & 0 & 0 & \boxed{0} \end{array}\right]$$

This is constat because x is in the span (there is a solution)

~~every~~ every column contains pivot as $v_1, \dots, v_d$ are L.I.

Thm: Given a subspace $V \subseteq \mathbb{R}^k$ all bases have the same size (the same # of vectors appear in any basis).

Def: The dimension of a subspace $V \subseteq \mathbb{R}^k$ is the number of vectors which appear in a basis for V.

Ex: $\dim(\mathbb{R}^K) = K$

why?

$e_1, \ldots, e_K$ is a basis.

Ex: the dimension of a line* in $\mathbb{R}^2$

is 1.

Ex: the dimension of a plane* in $\mathbb{R}^3$

is 2.

* = "through the origin."