Lecture II

Bases and Dimension.
General Problem

Given a subspace $V$ find the smallest set of vectors $v_1, \ldots, v_k$ such that

$$V = \text{span}(v_1, \ldots, v_k).$$

We saw last class that if one chose a spanning set containing too many vectors then there would be relations among those vectors i.e. expressions

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$$

would hold.
So if we want to find the smallest spanning set, we should look for vectors with few relations between them.

**Def:** Let \( \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n \).

We say \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) are linearly independent if the only expression for the zero vector of the form

\[
c_1 \mathbf{v}_1 + \ldots + c_k \mathbf{v}_k = \mathbf{0}
\]

is when \( c_i = 0 \) for all \( i \).
Examples:

\[ e_1 = [0], \quad e_2 = [1] \]

These vectors are l.i.

Why?

\[ 0 = c_1 e_1 + c_2 e_2 = [c_1] + [c_2] = [c_2]. \]

\[ c_1 = 0 \quad \text{and} \quad c_2 = 0. \]
More generally,

$e_1, \ldots, e_k$ standard basis vectors in $\mathbb{R}^k$ are L.I.

why?

$0 = c_1 e_1 + \ldots + c_k e_k$

$= \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \Rightarrow c_i = 0 \quad \text{for all } i$

Also, any subset of $\{e_1, \ldots, e_k\}$ is L.I.
Ex:

\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \]

are linearly independent.

**Why?** What are expressions of the form

\[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0} \]

\[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{0} \]

A \times \mathbf{v} = \mathbf{0}

We must relate \( \mathbf{x} \) by solving

\[ A\mathbf{x} = \mathbf{0} \quad \text{if} \quad A = \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 5 \end{bmatrix}. \]
Solve \( Ax = 0 \) by row reducibility.

\[
A = \begin{bmatrix} 1 & 7 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{row reduce} \\
\text{Keep this secret.}
\]

\[
\text{unpack} \\
\begin{align*}
c_1 &= 0 \\
c_2 &= 0
\end{align*}
\]

So \( v_1, v_2, v_3 \) are L.I.

To find relations between vectors, put vectors as columns of matrix and solve \( Ax = 0 \).
How do you determine if vectors are L.I.?

Then: The vectors $v_1, \ldots, v_k$ are L.I. if and only if

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_k \end{bmatrix}$$

has Kernel $\ker(A) = \{0\}$

(i.e., the only solution to $A\vec{x} = 0$ is $\vec{x} = 0$.)
i.e. if and only if

\[ \text{Ref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \]

\[ \cdots \]

\[ c_i = 0 \]

this matrix has pivots in every column.
Thus let $v_1, \ldots, v_k$ be linearly independent vectors, then every vector $b \in \text{Span}(v_1, \ldots, v_k)$ can be written as

$$b = c_1 v_1 + \ldots + c_k v_k$$

for exactly one set of $c_1, \ldots, c_k$.

i.e. this expression is unique.
(This generalizes that $\mathbf{b} \in \mathbb{R}^n$

$$\mathbf{b} = \left[ \begin{array}{c} c_1 \\ \vdots \\ c_n \end{array} \right] = c_1 \mathbf{v}_1 + \ldots + c_n \mathbf{v}_n$$

for some unique $c_1, \ldots, c_n$).

Why is this true?

If $\mathbf{b} = c_1 \mathbf{v}_1 + \ldots + c_k \mathbf{v}_k$

so $\mathbf{b} = c_1 \mathbf{v}_1 + \ldots + c_k \mathbf{v}_k$

then

$$\mathbf{0} = \mathbf{b} - \mathbf{b} = (c_1 - c_1') \mathbf{v}_1 + \ldots + (c_k - c_k') \mathbf{v}_k$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are L.I. all coeff are 0. so $c_1 - c_1' = 0$. 
\[ c_i = c'_i. \]

So, expressions for \( b \) above are the same.

Picture

\[ b = c_{i1} v_i + c_{i2} v_2 \]
Def: A basis for a subspace \( V \subseteq \mathbb{R}^n \) is an ordered collection of linearly independent vectors \( v_1, \ldots, v_k \in V \) such that

\[
V = \text{Span} (v_1, \ldots, v_k).
\]

By this, every element of \( V \) can be written uniquely as

\[
e_1 v_1 + \ldots + e_k v_k
\]

whenever \( v_1, \ldots, v_k \) is a basis.
Examples

1. \( e_1, \ldots, e_n \) in \( \mathbb{R}^n \) this is a basis for \( \mathbb{R}^n \).

2. \( e_2, e_1 \) is a basis for \( \mathbb{R}^2 \).
   Warning! Not the same basis as \( e_1, e_2 \). Overlap matters.

3. If \( v_1, v_2 \in \mathbb{R}^2 \) are not collinear then \( v_1, v_2 \) are a basis for \( \mathbb{R}^2 \).

4. \( [\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}], [\begin{smallmatrix} 7 \\ 5 \\ 3 \end{smallmatrix}] \) are a basis for \( V = \text{span} \{ [\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}], [\begin{smallmatrix} 7 \\ 5 \\ 3 \end{smallmatrix}] \} \)
Theorem: Given a subspace \( V \subseteq \mathbb{R}^k \), all bases for \( V \) have the same size (the same number of vectors appears in any basis).

Definition: The dimension of a subspace \( V \subseteq \mathbb{R}^k \) is the number of vectors which appear in any basis. It's denoted \( \text{dim}(V) \).
The dimension of $V$ is the smallest set of vectors which span $V$.

Examples:

1. $\dim (\mathbb{R}^n) = n$
   Why? Final basis $\{ e_1, \ldots, e_n \}$ Standard basis.

2. Dimension of a line through origin in $\mathbb{R}^n$. Dimension $= 1$
   If $v \neq 0$ and $v$ is in line $v$ is a basis.
(3) Dimensions of a plane in $\mathbb{R}^3$ through origin.

$\text{dim} = 2$

A basis is any 2 non-collinear vectors in $V$. 
Problem

Determine the dimension of

\[ V = \text{Span}\left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix} \right) \]

- Dimension is at most 3.
- Equals 3 if vectors above are l. i.
- Compute relations between vectors.

Form

\[ A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \]

Compute \( \text{Ker}(A) \) by row reducing \( A \).
Exercice compute $\text{Rref}(A)$.
and determine $\text{Ker}(A)$.

Cont. next class...