Lecture 10: Subspaces of $\mathbb{R}^k$

Last time we defined the span of a set of vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ to be the set

$$\text{Span}(v_1, \ldots, v_n) := \left\{ \sum_{i=1}^{n} c_i v_i : c_i \in \mathbb{R} \right\}$$

Let's see what these sets look like:

- $\text{Span}(\overrightarrow{0})$ is the origin.
- $\text{Span}(\overrightarrow{v})$ is a line through the origin containing $\overrightarrow{v}$. 

The origin.
The span of $v_1$ and $v_2$ is the plane through a containing $v_1$ and $v_2$, as long as $v_1$ and $v_2$ don't lie on the same line through $0$.

i.e. $\text{Span}(v_1, v_2) \neq \text{Span}(v_1)$ or $\text{Span}(v_2)$.

Any vector in the plane containing $v_1$ and $v_2$ and the origin is in the $\text{Span}(v_1, v_2)$.
The span of a set of vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ is a generalization of the notion of "line, plane, etc. through the origin containing $v_1, \ldots, v_n$.

Lines, planes, etc. through $0$ are generalized by the following definition:

**Def:** A subset $V \subseteq \mathbb{R}^k$ is called a **subspace** if

1. $\overrightarrow{0} \in V$.
2. If $v \in V$ then $cv \in V$ for all $c \in \mathbb{R}$.
3. If $v_1, v_2 \in V$, then $v_1 + v_2 \in V$.

**Examples:**

1. $\mathbb{R}^3 \subseteq \mathbb{R}^k$ is a subspace.
2. More generally, lines and planes through the origin are subspaces.
(2) If $V = \mathbb{R}^k$, then $V$ is a subspace of $\mathbb{R}^k$.

(3) If $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^k$, then $\operatorname{span}(v_1, \ldots, v_n)$ is a subspace.

(4) If $A$ is an $n \times m$ matrix (i.e. a linear transformation $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$), then

$$\ker(A) = \left\{ x \in \mathbb{R}^m \mid A x = 0 \right\}$$

is a subspace of $\mathbb{R}^m$ and

$$\operatorname{im}(A) = \left\{ b \in \mathbb{R}^n \mid b = A x \text{ for some } x \in \mathbb{R}^m \right\}$$

is a subspace of $\mathbb{R}^n$. 
The matrix $A$ from last time had kernel a plane through $0$ and image a line through $0$.

**Thm:** Every subspace $V \subseteq \mathbb{R}^k$ is equal to $\text{span}(v_1, \ldots, v_n)$ for some vectors $v_1, \ldots, v_n$. 
Warning: In general, there are many subsets \( \{ v_1, \ldots, v_n \} \) which span a subspace \( V \).

Ex: \( V = \mathbb{R}^2 \leq \mathbb{R}^2 \).

\[ \text{Span}(v_1, v_2) = V \text{ for all } v_1 \text{ and } v_2 \]

which are not co-linear.
Ex: $\text{Span } (v, 2v, 3v) = \text{Span } (v)$. 

You don't always need $n$ vectors to $\text{Span } (v_1, \ldots, v_n)$. 
General problem

Given a set of vectors $v_1, \ldots, v_n$, find the smallest subset of these vectors which span $\text{Span}(v_1, \ldots, v_n)$.

(Least example you only need 1 but were given 3).
Specific Problem

Let \( v_1, v_2, v_3, v_4, v_5 \)

\[
A = \begin{bmatrix}
1 & 2 & 2 & -5 & 6 \\
-1 & -2 & -1 & 1 & -1 \\
4 & 5 & -8 & 1 \\
3 & 6 & 1 & 5 & 7 \\
\end{bmatrix}
\]

We know that \( \text{Im}(A) \) is the span of the columns of \( A \).

Can we find fewer columns that \( \text{Span} \) \( \text{Im}(A) \)?
Solution:
Let's label the columns
$v_1, \ldots, v_5$

$\text{Im}(A) = \text{Span} \ (v_1, \ldots, v_5) \subseteq \mathbb{R}^4$

Idea: Scan across asking if at each step if
$\text{Span} \ (v_1, \ldots, v_i)$ is smaller or the same size as
$\text{Span} \ (v_1, \ldots, v_{i+1})$. 
let's do that.

\[ \text{Span}(v_1) \text{ it's a line.} \]

Is \[ \text{Span}(v_2, v_1) = \text{Span}(v_1) \]

is \( v_2 \) on the line \( \text{Span}(v_1) \)?

Yes, \( v_2 = 2v_1 \).

Row reduce

\[
\begin{bmatrix} v_1 & v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \end{bmatrix}
\]
Is \( \text{Span}(v_1, v_3) = \text{Span}(v_1) \)?

No.

Not a scalar multiple

\[
\begin{bmatrix}
  v_1 \\
v_3
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]

inconsistent.

So \( \text{Span}(v_1, v_3) \) is a plane.
Is \( \text{Span}(v_1, v_3) = \text{Span}(v_1, v_2, v_3, v_4) \)?

Find if there is an \( a, b \) such that

\[
ax_1 + bx_2 = x_4
\]

Set the following equations:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
v_1 & v_3 & v_4 & 1
\end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & -4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Yes, \( 3v_1 - 4v_3 = v_4 \).
Warm up:

Row reduce

\[
A = \begin{bmatrix}
1 & 2 & 2 \\
-1 & -2 & -5 \\
4 & 8 & 6 \\
3 & 6 & 5 \\
v_1 & v_2 & v_3
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 2 & 2 \\
0 & 1 & -4 \\
1 & 6 & 5 \\
v_1 & v_2 & v_3
\end{bmatrix}
\]

\[\text{Not in span of } v_1, v_2, v_3\]

\[3v_1 - 4v_2 \text{ Not in span of } v_1, v_2, v_3\]

\[\text{Pivot } \Rightarrow \text{ Span gets bigger.}\]

\[\text{Im}(A) = \text{Span } (v_1, v_3, v_5)\]
Thm: The image of a matrix $A$ is spanned by the columns which contain a pivot upon row reduction.