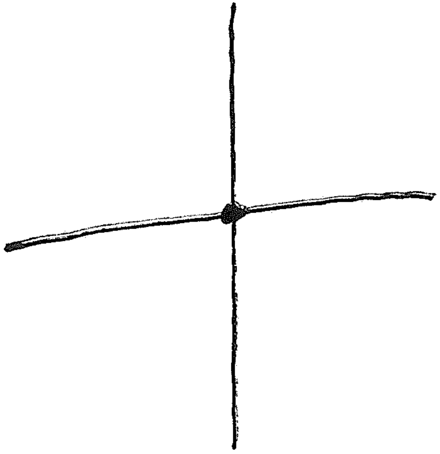
Lecture 10: Subspaces of $\mathbb{R}^K$

Last time we defined the <u>span</u> of a set of vectors $v_1, \dots, v_n \in \mathbb{R}^K$ to be the set

$$\text{Span}(v_1, \dots, v_n) := \{ c_1 v_1 + \dots + c_n v_n : c_i \in \mathbb{R} \}$$

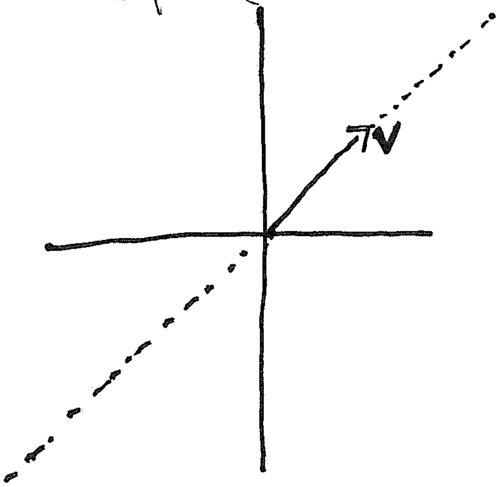
Let's see what these sets look like:

Span($\vec{0}$)

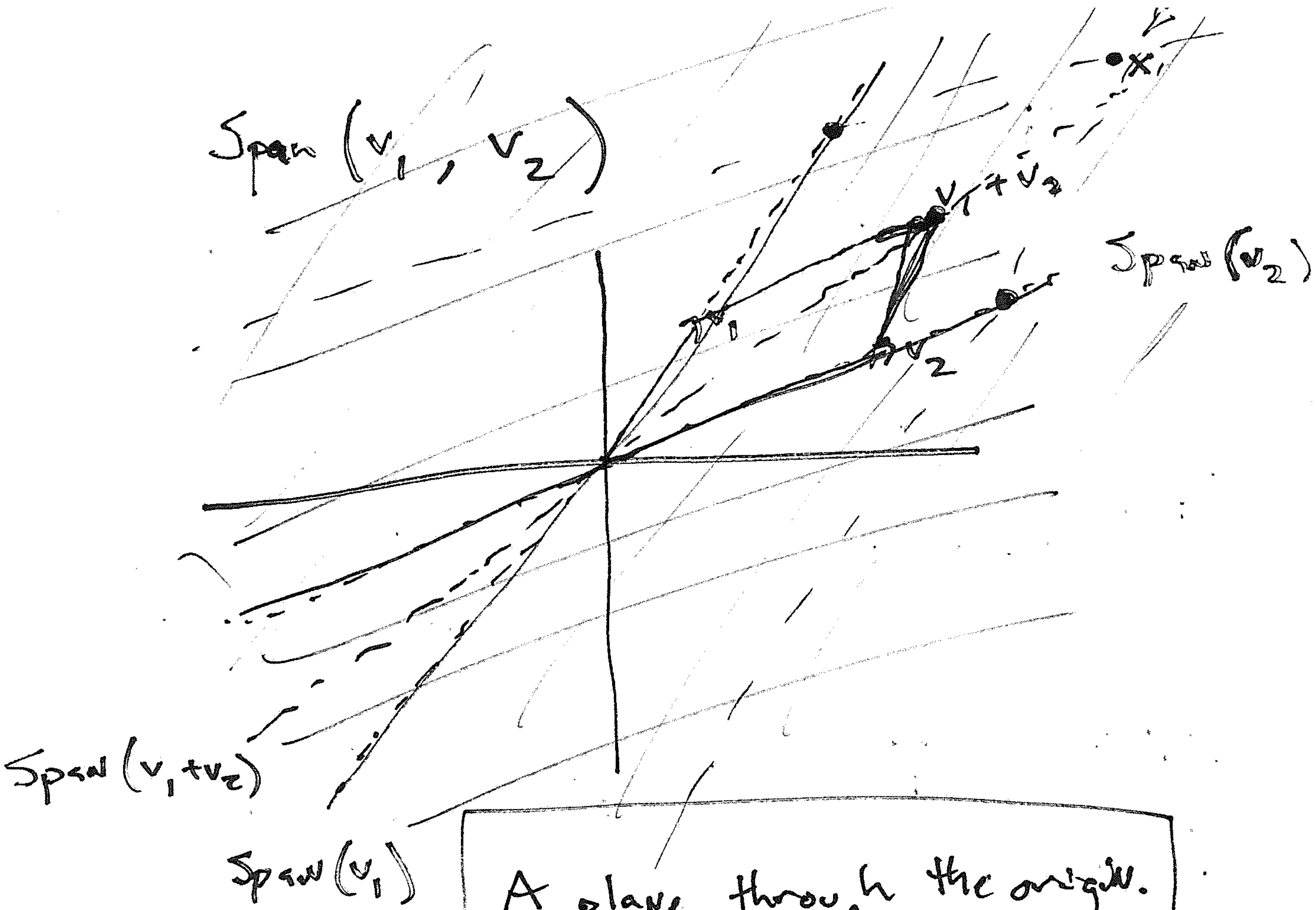


The origin.

Span($\vec{v}$) $v \neq 0$.



Line through the origin containg $\vec{v}$.



A plane through the origin.

Any vector in the plane containg v_1 and v_2 and the origin is in the span(v_1, v_2).

The span of v_1 and v_2 is the plane through 0 containg v_1 and v_2, as long as v_1 and v_2 don't lie on the same line through 0

i.e. $\text{span}(v_1, v_2) \neq \text{span}(v_1)$ or $\text{span}(v_2)$.

The span of a set of vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ is a generalization of the notion of "line, plane, etc. through the origin containing $v_1, \ldots, v_n$."

Lines, planes, etc. through 0 are generalized by the following definition

★ Def: A subset $V \subseteq \mathbb{R}^k$ is called a subspace if

① $\vec{0} \in V$.

② If $v \in V$ then $cv \in V$ for all $c \in \mathbb{R}$.

③ If $v_1, v_2 \in V$, then $v_1 + v_2 \in V$.

Examples: ① $\{\vec{0}\} \subseteq \mathbb{R}^k$ is a subspace.

(1') More generally lines and planes through the origin are subspaces.

② If $V = \mathbb{R}^K$, then V is a subspace of $\mathbb{R}^K$.

③ If $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^k$ then $\mathrm{span}(v_1, \ldots, v_n)$ is a subspace.

④ If A is an $n \times m$ matrix (i.e. a linear transformation $A: \mathbb{R}^m \longrightarrow \mathbb{R}^n$)

then

$$\mathrm{Ker}(A) = \{ x \in \mathbb{R}^m \mid Ax = 0 \}$$

is a subspace of $\mathbb{R}^m$ and

$$\mathrm{Im}(A) = \{ b \in \mathbb{R}^n \mid b = Ax \text{ for some } x \in \mathbb{R}^m \}$$

is a subspace of $\mathbb{R}^n$.

The matrix A from last time had kernel a plane through 0 and image a line through 0.

Thm: Every subspace $V \subseteq \mathbb{R}^k$ is equal to $\text{span}(v_1, \ldots, v_n)$ for some vectors $v_1, \ldots, v_n$.

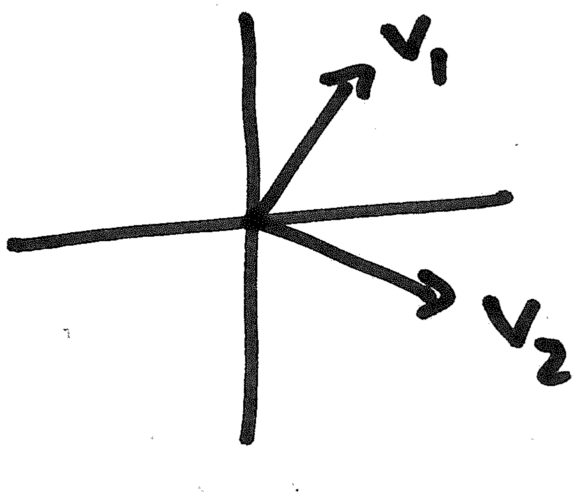
Warning: In general,

there are <u>MANY</u> subsets

$\{v_1, \ldots, v_n\}$ which span

a subspace V.

Ex: $V = \mathbb{R}^2 \subseteq \mathbb{R}^2$.



$\text{Span}(v_1, v_2) = V$

for all

v_1 and v_2

which are not colinear.

Ex: $\text{Span}(v, 2v, 3v)$

$= \text{Span}(v)$.

You don't always need n vectors to span

$$\text{Span}(v_1, \ldots, v_n).$$

General problem

Given a set of vectors $v_1, \ldots, v_n$, find the smallest subset of these vectors which span $\text{Span}(v_1, \ldots, v_n)$.

(Last example you only need 1 but were given 3).

Specific Problem

Let

$$A = \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{array}.$$

We know that Im(A) is the <u>span</u> of the columns of A.

Can we find fewer columns that span Im(A)?

Solution:

Let's label the columns $v_1, \dots, v_5$.

$$\mathrm{Im}(A) = \mathrm{Span}(v_1, \dots, v_5)$$
$$\subseteq \mathbb{R}^4$$

Idea: Scan across asking if at ~~each~~ each step if $\mathrm{Span}(v_1, \dots, v_i)$ is smaller or the same size as $\mathrm{Span}(v_1, \dots, v_{i+1})$.

Let's do that.

$\text{Span}(v_1)$ it's a line.

……

Is $\text{Span}(v_2, v_1) \overset{?}{=} \text{Span}(v_1)$

is v_2 on the line

$\text{Span}(v_1)$?

Yes, $v_2 = 2v_1$.

Row reduce

$$\left[\, v_1 \;\vdots\; v_2 \,\right] \Longrightarrow \left[\begin{array}{c:c} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right].$$

Is $\text{Span}(v_1, v_3) \overset{?}{=} \text{Span}(v_1)$.

No.

Not a Scalar multiple

$$\left[\begin{array}{c:c} v_1 & v_3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{c:c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

inconsistent.

So $\text{Span}(v_1, v_3)$ is a plane.

Is $\text{Span}(v_1, v_3) \overset{?}{=} \text{Span}(v_1, v_2, v_3, v_4)$?

Find if there is an a, b such that

$$a v_1 + b v_3 = v_4$$

~~Solve~~ Find solutions to

$$\left[\begin{array}{cc|c} | & | & | \\ v_1 & v_3 & v_4 \\ | & | & | \end{array}\right].$$

$$\Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Yes, $3v_1 - 4v_3 = v_4$.

Warm up:

Row reduce

$$A = \begin{array}{c} \left[\begin{array}{ccccc} \boxed{1} & 2 & \boxed{2} & -5 & \boxed{6} \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{array}\right]. \\ \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \\ & 2v_2 & \text{Not in span} & 3v_1 - 4v_3 & \text{Not in span of } v_1, v_3. \end{array} \end{array}$$

$$\text{Rref}(A) = \begin{bmatrix} \boxed{1} & \boxed{2} & 0 & +3 & 0 \\ 0 & 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Pivot $\Rightarrow$ Span gets bigger.

$\text{Im}(A) = \text{Span}(v_1, v_3, v_5).$

Thm: The image of a matrix A is spanned by the columns which contain a pivot upon row reduction.