Homework solutions   Due 4/12

Section 7.1

Find all eigenvalues and eigenvectors, also find an eigenbasis if you can.

15. Reflection about a line L in $\mathbb{R}^2$.

This has two eigenvalues $\begin{cases} +1 & \text{vectors on the line} \\ -1 & \text{vectors vertical to the line.} \end{cases}$

16. Rotations through an angle of $180°$.

It has only one eigenvalue -1, and all vectors in $\mathbb{R}^2$ are its eigenvectors

18. Reflection about a plane V in $\mathbb{R}^3$.

This has two eigenvalues $\begin{cases} +1 & \text{eigenvectors are vectors on the plane. so they form a 2-dim'l space} \\ -1 & \text{eigenvectors are the normal vector to the plane. 1-dim'l.} \end{cases}$

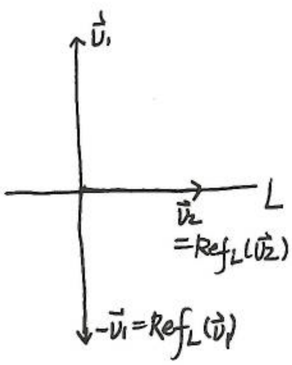
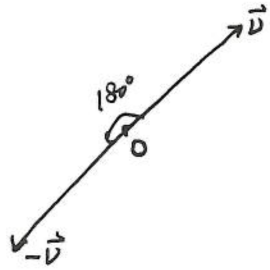


Figure 15.



(just Reflection about the point o).
Figure 16

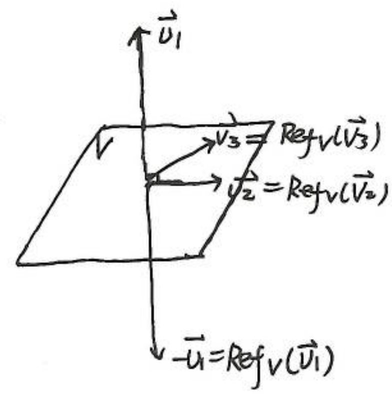


Figure 18.

Find an eigenbasis for each of the matrices, and thus diagonalize them.

56. $A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$. i.e. reflection about the line $L = \text{Span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

As in Ex 15 above, we know A has two eigenvalues 1 (eigen vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$) & -1 (eigenvectors $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$).

So we have an eigenbasis $(\vec{v}_1, \vec{v}_2) = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ of A. $\Rightarrow$ if we let $S = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$.

then we can diagonalize A by $S^{-1}AS = B$. where $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (you can check this!).

↑ we use theorem 7.1.3 here!

62. $A = \frac{1}{14}\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$. representing the orthogonal projection onto a plane E.

Solving the linear system $A\vec{x} = 0$. we get $\vec{x} = k\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $k \in \mathbb{R}$. so we know A sends vectors on the line $L = \text{Span}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to 0. By property of orthogonal projection, we then know $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

is just the ~~eigen~~ normal vector of the plane E. $\Rightarrow$ the linear transformation given by A has two eigenvalues 0 & 1. and the corresponding eigenvectors are:

0: eigenvectors $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot k$. $k\in\mathbb{R}$. i.e. vectors normal to the plane E.

1: eigenvectors are the vectors on the plane $E: x+2y+3z=0$, so it has a basis $\left\{\begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\-1\end{bmatrix}\right\}$.

Combining these. A has an eigen basis $\begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$. $\Rightarrow$ if we set $S=\begin{bmatrix}2&3&1\\-1&0&2\\0&-1&3\end{bmatrix}$.

then $S^{-1}AS=B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$. Is the diagonalization of A.

Section 7.2

Find the algebraic multiplicities of all real eigenvalues of the following matrices.

2. $\begin{bmatrix}2&0&0&0\\2&1&0&0\\2&1&2&0\\2&1&2&1\end{bmatrix}$ characteristic polynomial $=\det(A-\lambda I_4)=\det\begin{bmatrix}2-\lambda&0&0&0\\2&1-\lambda&0&0\\2&1&2-\lambda&0\\2&1&2&1-\lambda\end{bmatrix}$

$=(\lambda-2)^2(\lambda-1)^2$. So it has two eigenvalues 1, 2 both having multiplicity 2.

10. $\begin{bmatrix}-3&0&4\\0&-1&0\\-2&7&3\end{bmatrix}$ $A-\lambda I_3=\begin{bmatrix}-3-\lambda&0&4\\0&-1-\lambda&0\\-2&7&3-\lambda\end{bmatrix}$. use Laplace explansion on the second row $=(-1-\lambda)\det\begin{bmatrix}-3-\lambda&4\\-2&3-\lambda\end{bmatrix}$

$=-(\lambda+1)(\lambda^2-9+8)=-(\lambda+1)(\lambda+1)(\lambda-1)=-(\lambda+1)^2(\lambda-1)$. So it has two eigenvalues, -1 with multiplicity 2 & 1 with multiplicity 1.

12. $\begin{bmatrix}2&-2&0&0\\1&-1&0&0\\0&0&3&-4\\0&0&2&-3\end{bmatrix}$ $A-\lambda I_4=\begin{bmatrix}2-\lambda&-2&0&0\\1&-1-\lambda&0&0\\0&0&3-\lambda&-4\\0&0&2&-3-\lambda\end{bmatrix}$. this is a block matrix, so $\det(A-\lambda I_4)=$

$\det\begin{bmatrix}2-\lambda&-2\\+1&-1-\lambda\end{bmatrix}\cdot\det\begin{bmatrix}3-\lambda&-4\\2&-3-\lambda\end{bmatrix}=\left((\lambda+1)(\lambda-2)+2\right)\cdot\left(\lambda^2-9+8\right)=\lambda(\lambda-1)\cdot(\lambda+1)(\lambda-1)$

Hence A has 3 eigenvalues $0, \pm1$. both 0 & -1 have multiplicity 1, and 1 has multiplicity 2.

15. Consider $A=\begin{bmatrix}1&k\\1&1\end{bmatrix}$. for which k does A have two distinct real eigenvalues? or no eigenvalues?

$\det(A-\lambda I_2)=\det\begin{pmatrix}1-\lambda & k\\ 1 & 1-\lambda\end{pmatrix}=(\lambda-1)^2-k=\lambda^2-2\lambda+1-k$. this quadratic polynomial has discriminant $=b^2-4ac=2^2-4\cdot1\cdot(1-k)=4-4(1-k)=4k$. Hence we have:

$\begin{cases}\text{A has two distinct real eigenvalues} \Leftrightarrow \text{discriminant}=4k>0 \Leftrightarrow k>0\\ \text{A has no real eigenvalues} \Leftrightarrow \text{discriminant}=4k<0 \Leftrightarrow k<0.\end{cases}$

16. consider $A=\begin{bmatrix}a & b\\ b & c\end{bmatrix}$, where a.b.c nonzero constants. For which a.b.c does A have two distinct eigenvalues?

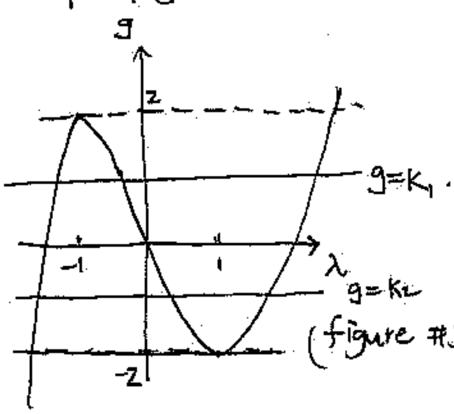
$\det(A-\lambda I_2)=\det\begin{bmatrix}a-\lambda & b\\ b & c-\lambda\end{bmatrix}=(\lambda-a)(\lambda-c)-b^2=\lambda^2-(a+c)\lambda+ac-b^2$. its discriminant $=b^2-4ac$ $=(a+c)^2-4(ac-b^2)=a^2+2ac+c^2-4ac+4b^2=(a-c)^2+(2b)^2$. Note this is always nonnegative, and equals to zero if and only if $a-c=b=0$. i.e. $a=c$, $b=0$. So except for the case $a=c$. $b=0$. A has two distinct eigenvalues. but $b\neq0$ by assumption, so A always has two distinct eigenvalues.

32. Consider $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ k & 3 & 0\end{bmatrix}$ k an arbitrary constant. For which values of k does A have three distinct real eigenvalues? For which k does A have two distinct eigenvalues?

$\det(A-\lambda I_3)=\det\begin{bmatrix}-\lambda & 1 & 0\\ 0 & -\lambda & 1\\ k & 3 & -\lambda\end{bmatrix}\overset{\text{Laplace expansion on first row}}{=}-\lambda\det\begin{bmatrix}-\lambda & 1\\ 3 & -\lambda\end{bmatrix}-1\cdot\det\begin{bmatrix}0 & 1\\ k & -\lambda\end{bmatrix}=-\lambda(\lambda^2-3)+k=-(\lambda^3-3\lambda-k)$.

Let $g(\lambda)=\lambda^3-3\lambda$. $g'(\lambda)=3\lambda^2-3=3(\lambda+1)(\lambda-1)\begin{cases}\geq0 \text{ when } \lambda>1 \text{ or } \lambda<-1\\ \leq0 \text{ when } -1<\lambda<1.\end{cases}$ so $\begin{cases}1 \text{ is local minimum}\\ -1 \text{ is local maximum}\end{cases}$

Graph of g looks like



(figure #32).

Now we draw the horizontal lines $g=k$ for different k. We can see it has three intersection points with graph of g iff $k\in(-2,2)$, and two intersection points when $k=\pm2$.

To conclude, A has three distinct eigenvalues $\Leftrightarrow k\in(-2,2)$

A has two distinct eigenvalues when $k=\pm2$.

Section 7.3

Find all eigenvalues of following matrices, then a basis of each eigenspace. then diagonalize A if you can.

In general, follow theorem 7.3.7.

4. $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$. $\det(A-\lambda I_2) \overset{\det}{=} \begin{bmatrix} -\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2-2\lambda+1=(\lambda-1)^2$. the only eigenvalue is 1.

Now we find $\ker(A-I_2)=\ker\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ Solving the system $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\vec{x}=0$. we have

$\ker(A-I_2)=\text{Span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. i.e. the eigenspace of 1 is $\text{Span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\dim \neq 2$. the matrix is not diagonalizable.

6. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\det(A-\lambda I_2)=\det\begin{bmatrix} 2-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix}=\lambda^2-7\lambda+10-12=\lambda^2-7\lambda-2$ the eigenvalues are

$\frac{7\pm\sqrt{49+8}}{2}=\frac{7\pm\sqrt{57}}{2}$. For $\lambda_1=\frac{7+\sqrt{57}}{2}$, $A-\lambda_1 I_2=\begin{bmatrix} \frac{-3-\sqrt{57}}{2} & 3 \\ 4 & \frac{3-\sqrt{57}}{2} \end{bmatrix}$. $\ker(A-\lambda_1 I_2)=\text{Span}\begin{bmatrix} 3 \\ \frac{3+\sqrt{57}}{2} \end{bmatrix}$

Similarly, $\ker(A-\lambda_2 I_2)=\ker\begin{bmatrix} \frac{-3+\sqrt{57}}{2} & 3 \\ 4 & \frac{-3-\sqrt{57}}{2} \end{bmatrix}=\text{Span}\begin{bmatrix} 3 \\ \frac{3-\sqrt{57}}{2} \end{bmatrix}$. these are the eigenspaces.

and the sum of their dimension is 2 $\Rightarrow$ A is diagonalizable. if $S=\begin{bmatrix} 3 & 3 \\ \frac{3+\sqrt{57}}{2} & \frac{3-\sqrt{57}}{2} \end{bmatrix}$

then $S^{-1}AS=\begin{bmatrix} \frac{7+\sqrt{57}}{2} & 0 \\ 0 & \frac{7-\sqrt{57}}{2} \end{bmatrix}$.

14. $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. $\det(A-\lambda I_3)=\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ -5 & -\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix} \overset{\text{Laplace expansion on first row}}{=} (1-\lambda)\cdot\det\begin{bmatrix} -\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}=-\lambda(1-\lambda)^2$, $\Rightarrow$ A has two

eigenvalues 0&1. For $\lambda=0$. the eigenspace $=\ker(A-0\cdot I_3)=\ker(A)=\text{Span}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (just solve $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$) for $\lambda=1$. the eigenspace $=\ker(A-I_3)=\ker\begin{pmatrix} 0 & 0 & 0 \\ -5 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}=\text{Span}\left(\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix},\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right)$.

So if we set $S=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 5 & 1 & 0 \end{bmatrix}$. $S^{-1}AS=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is its diagonalization.

18. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. $\det(A-\lambda I_4)=\det\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}=\lambda^2(\lambda-1)^2$. has two eigenvalues $\lambda=0$ or 1.

for $\lambda=0$. eigenspace $=\ker(A)=\ker\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\text{Span}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

for $\lambda=1$, eigenspace $=\ker(A-\cancel{\lambda}I)=\ker\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}=\text{Span}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. the sum of their dimension

$=3$. not equal to 4. so this matrix can't be diagonalized.

20. $A=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. $\det(A-\lambda I)=\det\begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix} \overset{\text{Laplace expansion on second column.}}{=} (1-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}=(1-\lambda)(\lambda^2-2\lambda+1-1)$

$= (1-\lambda)\lambda(\lambda-2)$. So A has three eigenvalues $0, 1, 2$.

$\lambda=0$. eigenspace $= \ker A = \ker \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \mathrm{Span}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (dim $=1$)

$\lambda=1$ eigenspace $= \ker A-I_3 = \ker \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \mathrm{Span}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (dim $=1$).

$\lambda=2$ eigenspace $= \ker(A-2I) = \ker \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \mathrm{Span}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. (dim $=1$)

the dimensions add up to 3. so A is diagonalizable: if $S = \begin{array}{c} \begin{matrix} \lambda=0 & \lambda=1 & \lambda=2 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \end{array}$

then $S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

52. Find the characteristic polynomial of the $n\times n$ matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \end{bmatrix}.$$

Let $A_n = \det \begin{bmatrix} -\lambda & 0 & 0 & \cdots & 0 & a_0 \\ 1 & -\lambda & 0 & \cdots & 0 & a_1 \\ 0 & 1 & -\lambda & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1}-\lambda \end{bmatrix} = \det(A-\lambda I_n)$. use Laplace expansion on the first column,

we have $A_n = -\lambda \underbrace{\det \begin{bmatrix} -\lambda & 0 & \cdots & 0 & a_1 \\ 1 & -\lambda & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1}-\lambda \end{bmatrix}}_{B} - \det \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & -\lambda & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1}-\lambda \end{bmatrix} \overset{\text{Laplace expansion on first row}}{=} a_0 \cdot (-1)^{n-1+1} \det \begin{bmatrix} 1 & -\lambda & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & -\lambda \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$\rightarrow (n-1)\times(n-1)$ matrix

$= (-1)^n a_0 \quad (*)$.

using $(*)$ and induction, we can then prove $A_n = (-1)^n(\lambda^n - a_{n-1}\lambda - \cdots - a_0)$. ①

the base case $n=1$ is easy. if we have ① for $n-1$. the det of B in $(*)$

$= (-1)^{n-1}(\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_1)$. plug this in $(*)$. we have

$A_n = (-\lambda)\cdot(-1)^{n-1}(\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_1) + (-1)^{n+1} a_0 = (-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda) + (-1)^{n+1} a_0$

$= (-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_0)$ as claimed.