Homework Math 201

Apr. 05 solutions

Section 6.2

(C.f. theorem 6.2.3 for general results)

2.4.6.8: use Gaussian elimination to find the determinants of the following:

2. $A=\begin{bmatrix}1&2&3\\1&6&8\\-2&-4&0\end{bmatrix}\xrightarrow[\text{III}+2\cdot\text{I}]{\text{II}-\text{I}}\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}\xrightarrow[\frac{1}{6}\cdot\text{III}]{\frac{1}{4}\cdot\text{II}}\begin{bmatrix}1&2&3\\0&1&\frac{5}{4}\\0&0&1\end{bmatrix}\xrightarrow[\text{II}-\frac{5}{4}\text{III}]{}\begin{bmatrix}1&2&3\\0&1&0\\0&0&1\end{bmatrix}$

determinant doesn't change at this step

determinant / 24 here

doesn't change

$\xrightarrow[\text{I}-3\cdot\text{III}]{\text{I}-2\cdot\text{II}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\longrightarrow$ identity matrix with determinant 1. Hence $\det A=24$.

4. $A=\begin{bmatrix}1&-1&2&-2\\-1&2&1&6\\2&1&14&10\\-2&6&10&33\end{bmatrix}\xrightarrow[\substack{\text{III}-2\cdot\text{I}\\ \text{IV}+2\cdot\text{I}}]{\text{II}+\text{I}}\begin{bmatrix}1&-1&2&-2\\0&1&3&4\\0&3&10&14\\0&4&14&29\end{bmatrix}\xrightarrow[\text{IV}-4\cdot\text{II}]{\text{III}-3\cdot\text{II}}\begin{bmatrix}1&-1&2&-2\\0&1&3&4\\0&0&1&2\\0&0&2&13\end{bmatrix}\xrightarrow{\text{IV}-2\cdot\text{III}}\begin{bmatrix}1&-1&2&-2\\0&1&3&4\\0&0&1&2\\0&0&0&9\end{bmatrix}$

doesn't change here

doesn't change

doesn't change

the last matrix has determinant 9. hence $\det A=9$.

6. $A=\begin{bmatrix}1&1&1&1\\1&1&4&4\\1&-1&2&-2\\1&-1&8&-8\end{bmatrix}\xrightarrow[\substack{\text{III}-\text{I}\\ \text{IV}-\text{I}}]{\text{II}-\text{I}}\begin{bmatrix}1&1&1&1\\0&0&3&3\\0&-2&1&-3\\0&-2&7&-9\end{bmatrix}\xrightarrow[\text{III}\leftrightarrow\text{IV}]{\text{II}\leftrightarrow\text{III}}\begin{bmatrix}1&1&1&1\\0&-2&1&-3\\0&-2&7&-9\\0&0&3&3\end{bmatrix}\xrightarrow{\text{III}-\text{II}}\begin{bmatrix}1&1&1&1\\0&-2&1&-3\\0&0&6&-6\\0&0&3&3\end{bmatrix}$

determinant doesn't change b/c we performed two row swaps here

$\xrightarrow{\text{IV}-\frac{1}{2}\text{III}}\begin{bmatrix}1&1&1&1\\0&-2&1&-3\\0&0&6&-6\\0&0&0&6\end{bmatrix}$. note during this process the determinant never changes. hence

$\det A=$ det of the last matrix $=1\cdot(-2)\cdot6\cdot6=-72$.

8. $A=\begin{bmatrix}0&0&0&0&2\\1&0&0&0&3\\0&1&0&0&4\\0&0&1&0&5\\0&0&0&1&6\end{bmatrix}\xrightarrow[\text{row to the fifth}]{\text{put the first}}\begin{bmatrix}1&0&0&0&3\\0&1&0&0&4\\0&0&1&0&5\\0&0&0&1&6\\0&0&0&0&2\end{bmatrix}\rightarrow$ has determinant $=1\cdot1\cdot1\cdot1\cdot2=2$

hence $\det A=2$.

fifth, the determinant doesn't change b/c we performed 4 row swaps in total! (and $(-1)^4=1$)

$\text{I}\leftrightarrow\text{II},\ \text{II}\leftrightarrow\text{III},\ \text{III}\leftrightarrow\text{IV},\ \text{IV}\leftrightarrow\text{V}$.

40. If A is an orthogonal matrix, what are the possible values of $\det A$?

Recall A is said to be orthogonal if & only if $A \cdot A^T = I$ ①. since $\det A = \det A^T$, taking determinants on both sides of ①, we have $\det^2 A = 1$. $\Rightarrow \det A = \pm 1$.

Both ± 1 can be taken. for example. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is orthogonal with $\det = 1$; $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ orthogonal with $\det = -1$.

44. (The cross product in $\mathbb{R}^n$).

a. When is $\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n} = 0$? Give your answer in terms of linear independence.

$\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n} = 0 \Leftrightarrow \vec{x} \cdot (\vec{v_2} \times \cdots \times \vec{v_n}) = 0$ for all $\vec{x} \in \mathbb{R}^n \Leftrightarrow$

$\det \begin{bmatrix} | & | & & | \\ \vec{x} & \vec{v_2} & \cdots & \vec{v_n} \\ | & | & & | \end{bmatrix} = 0$ for all $\vec{x} \in \mathbb{R}^n \Leftrightarrow \vec{x}, \vec{v_2}, \cdots, \vec{v_n}$ linearly dependent for all $\vec{x}$

$\Leftrightarrow \vec{v_2}, \cdots, \vec{v_n}$ are linearly dependent.

b. Find $\vec{e_2} \times \vec{e_3} \times \cdots \times \vec{e_n}$.

the i-th component of this vector is $\vec{e_i} \cdot (\vec{e_2} \times \cdots \times \vec{e_n}) = \det \begin{bmatrix} | & | & & | \\ \vec{e_i} & \vec{e_2} & \cdots & \vec{e_n} \\ | & | & & | \end{bmatrix}$.

if $2 \le i \le n$. this matrix then have 2 same columns $\vec{e_i}$, hence has 0-~~component~~ determinant;

if $i = 1$. this matrix is just the identity matrix, hence has determinant $= 1$.

$\Rightarrow \vec{e_2} \times \vec{e_3} \times \cdots \vec{e_n} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{e_1}$.

c. Show that $\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n}$ is orthogonal to all $\vec{v_i}$, $i = 2, \cdots, n$.

$\vec{v_i} \cdot (\vec{v_2} \times \cdots \times \vec{v_n}) = \det \begin{bmatrix} | & | & & | \\ \vec{v_i} & \vec{v_2} & \cdots & \vec{v_n} \\ | & | & & | \end{bmatrix} = 0$ (b/c in this matrix, the first column & the i-th are the same ($\vec{v_i}$). it has determinant 0).

d. what is the relationship between $\vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n}$ and $\vec{v_3} \times \vec{v_2} \times \cdots \times \vec{v_n}$?

the i-th component of the two vectors are respectively $\vec{e_i} \cdot (\vec{v_2} \times \cdots \times \vec{v_n})$ & $\vec{e_i} \cdot (\vec{v_3} \times \vec{v_2} \times \cdots \times \vec{v_n})$.

i.e. $\det \begin{bmatrix} | & | & & | \\ \vec{e_i} & \vec{v_2} & \cdots & \vec{v_n} \\ | & | & & | \end{bmatrix}$ & $\det \begin{bmatrix} | & | & | & & | \\ \vec{e_i} & \vec{v_3} & \vec{v_2} & \cdots & \vec{v_n} \\ | & | & | & & | \end{bmatrix}$. Note $\det \begin{bmatrix} | & | & | & & | \\ \vec{e_i} & \vec{v_3} & \vec{v_2} & \cdots & \vec{v_n} \\ | & | & | & & | \end{bmatrix} = -\det \begin{bmatrix} | & | & & | \\ \vec{e_i} & \vec{v_2} & \cdots & \vec{v_n} \\ | & | & & | \end{bmatrix}$

b/c we swapped the second & third column $\Rightarrow \vec{v_2} \times \vec{v_3} \times \cdots \times \vec{v_n} = -\vec{v_3} \times \vec{v_2} \times \cdots \times \vec{v_n}$.

(every component of the latter is minus the former).

e. Express $\det\begin{bmatrix}\vec{v_2}\times\cdots\times\vec{v_n} & \vec{v_2} & \vec{v_3} & \cdots & \vec{v_n}\end{bmatrix}$ in terms of $\|\vec{v_2}\times\cdots\times\vec{v_n}\|$.

Note $\det\begin{bmatrix}\vec{v_2}\times\cdots\times\vec{v_n} & \vec{v_2} & \cdots & \vec{v_n}\end{bmatrix} = (\vec{v_2}\times\cdots\times\vec{v_n})\cdot(\vec{v_2}\times\cdots\times\vec{v_n}) = \|\vec{v_2}\times\cdots\times\vec{v_n}\|^2$.

f. How do we know the cross product of two vectors in $\mathbb{R}^3$ defined here is the same as the standard cross product in $\mathbb{R}^3$?

Suppose $\vec{v}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$, $\vec{w}=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ recall in appendix A. the cross product $\vec{v}\times\vec{w}$ is given in coordinates by $\vec{v}\times\vec{w}=\begin{bmatrix}v_2w_3-v_3w_2\\v_3w_1-v_1w_3\\v_1w_2-v_2w_1\end{bmatrix}$. ① Now from our definition, the i-th component of $\vec{v}\times\vec{w}$ should be $\vec{e_i}\cdot(\vec{v}\times\vec{w}) = \det\begin{bmatrix}\vec{e_i} & \vec{v} & \vec{w}\end{bmatrix} = \det\begin{bmatrix}1 & v_1 & w_1\\0 & v_2 & w_2\\0 & v_3 & w_3\end{bmatrix}$ ② (when $i=1$) $=\det\begin{bmatrix}v_2 & w_2\\v_3 & w_3\end{bmatrix}$ $=v_2w_3-v_3w_2$. Similarly in cases $i=2$ or 3. we can check the formulas ① & ② give the same i-th component of $\vec{v}\times\vec{w}$. Hence the two definitions coincide.

46. Given a.b.c.d.e.f. s.t. $\det\underbrace{\begin{bmatrix}a & 1 & d\\b & 1 & e\\c & 1 & f\end{bmatrix}}_{=A}=7$ ① & $\det\underbrace{\begin{bmatrix}a & 1 & d\\b & 2 & e\\c & 3 & f\end{bmatrix}}_{=B}=11$. ②.

a. find $\det\begin{bmatrix}a & 3 & d\\b & 3 & e\\c & 3 & f\end{bmatrix}$ this matrix is just multiplying the second column of A by 3.

hence $\det\begin{bmatrix}a & 3 & d\\b & 3 & e\\c & 3 & f\end{bmatrix}=3\det A=21$.

b. find $\det\begin{bmatrix}a & 3 & d\\b & 5 & e\\c & 7 & f\end{bmatrix}$.

Note $\det\begin{bmatrix}a & 3 & d\\b & 5 & e\\c & 7 & f\end{bmatrix} = -\det\begin{bmatrix}3 & a & d\\5 & b & e\\7 & c & f\end{bmatrix} = -3\det\begin{bmatrix}b & e\\c & f\end{bmatrix}+5\det\begin{bmatrix}a & d\\c & f\end{bmatrix}-7\det\begin{bmatrix}a & d\\b & e\end{bmatrix}$.

from conditions ① & ②, we have

$\det\begin{bmatrix}a & 1 & d\\b & 1 & e\\c & 1 & f\end{bmatrix} = -\det\begin{bmatrix}1 & a & d\\1 & b & e\\1 & c & f\end{bmatrix} = -\det\begin{bmatrix}b & e\\c & f\end{bmatrix}+\det\begin{bmatrix}a & d\\c & f\end{bmatrix}-\det\begin{bmatrix}a & d\\b & e\end{bmatrix}=+7$ ③

$\det\begin{bmatrix}a & 1 & d\\b & 2 & e\\c & 3 & f\end{bmatrix} = -\det\begin{bmatrix}b & e\\c & f\end{bmatrix}+2\det\begin{bmatrix}a & d\\c & f\end{bmatrix}-\det\begin{bmatrix}a & d\\b & e\end{bmatrix}=+11$. ④.

③ + 2·④, we get $-3\det\begin{bmatrix}b & e\\c & f\end{bmatrix}+5\det\begin{bmatrix}a & d\\c & f\end{bmatrix}-7\det\begin{bmatrix}a & d\\b & e\end{bmatrix}=29$. but the LH is just $\det\begin{bmatrix}a & 3 & d\\b & 5 & e\\c & 7 & f\end{bmatrix}$. hence we have $\det\begin{bmatrix}a & 3 & d\\b & 5 & e\\c & 7 & f\end{bmatrix}=29$.

66. a. Find a formula expressing d_n in terms of d_{n-1} & d_{n-2}. for $n \geq 3$.

$$M_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We can then use Laplace expansion to M_n to get: (applying to the first column)

$$\det M_n = a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots + (-1)^{n+1} \det A_{n1} = \det A_{11} - \det A_{21} \quad (*)$$ (note $a_{i1} = 0$ when $i \geq 3$).

now $\underset{\substack{\downarrow \\ (n-1)\times(n-1) \\ \text{matrix}}}{A_{11}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix} = M_{n-1}$. so $\det A_{11} = \det M_{n-1} = d_{n-1}$;

$\underset{\substack{\downarrow \\ (n-1)\times(n-1)}}{A_{21}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & M_{n-2} \end{bmatrix}$. so $\det A_{21} = \det M_{n-2} = d_{n-2}$.

plug in these two back in (*). we see $d_n = \det M_n = d_{n-1} - d_{n-2}$.

b. $d_1 = \det M_1 = \det [1] = 1$; $d_2 = \det M_2 = \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$.

$d_3 = d_2 - d_1 = -1$; $d_4 = d_3 - d_2 = -1$; $d_5 = d_4 - d_3 = 0$; $d_6 = d_5 - d_4 = 1$; $d_7 = d_6 - d_5 = 1$;

$d_8 = d_7 - d_6 = 0$.

c. what is the relationship between d_n & d_{n+3}? d_n & d_{n+6}?

from our computation in part (b). we can make the guess $d_n = -d_{n+3} = d_{n+6}$.

We only need to check $d_n = -d_{n+3}$. then $d_{n+6} = -d_{n+3} = d_n$ follows.

note $d_n = d_{n-1} - d_{n-2}$. we have $d_{n+3} = d_{n+2} - d_{n+1} = (d_{n+1} - d_n) - d_{n+1} = -d_n$.

d. Find d_{100}.

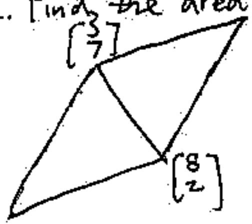
from our result in part (c). $d_{100} = d_{94} = d_{88} = \cdots = d_4 = -1$. ($d_4 = -1$ from part (b)).

Section 6.3.

1. Find the area of the parallelogram defined by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ & $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

recall this area $= |\det A| = \left| \det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} \right| = |3 \cdot 2 - 8 \cdot 7| = 50$.

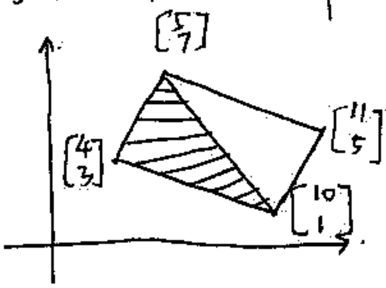
2. Find the area of the triangle defined by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ & $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.



area of the triangle $= \frac{1}{2}$ area of the parallelogram enclosed by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ & $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$= \frac{1}{2} \cdot 50 = 25$. (uses exercise 1 above)

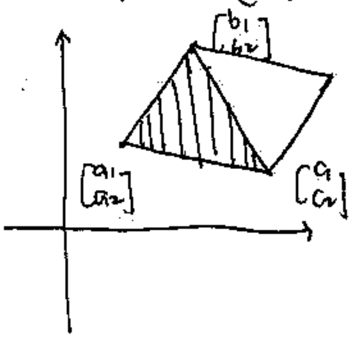
3. Find the area of the triangle below.



area of triangle $= \frac{1}{2}$ area of the parallelogram enclosed by the two vectors $\begin{bmatrix}1\\4\end{bmatrix}$ & $\begin{bmatrix}6\\-2\end{bmatrix}$

$$= \frac{1}{2}\left|\det\begin{bmatrix}1 & 6\\4 & -2\end{bmatrix}\right| = \frac{1}{2}\left|1\cdot(-2) - 4\cdot 6\right| = \frac{1}{2}\cdot 26 = 13.$$

4. Consider the area of the triangle with vertices $\begin{bmatrix}a_1\\a_2\end{bmatrix}$. $\begin{bmatrix}b_1\\b_2\end{bmatrix}$. $\begin{bmatrix}c_1\\c_2\end{bmatrix}$. Express A in terms of $\det\begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\1 & 1 & 1\end{bmatrix}$.



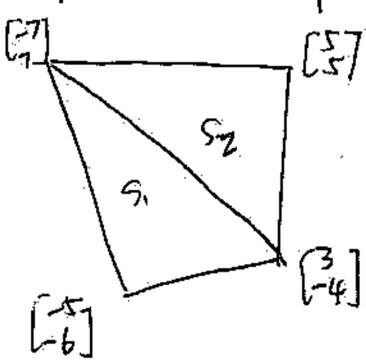
area of the triangle $= \frac{1}{2}$ area of the parallelogram enclosed by vectors $\begin{bmatrix}b_1-a_1\\b_2-a_2\end{bmatrix}$ & $\begin{bmatrix}c_1-a_1\\c_2-a_2\end{bmatrix}$.

$$= \frac{1}{2}\left|\det\begin{bmatrix}b_1-a_1 & c_1-a_1\\b_2-a_2 & c_2-a_2\end{bmatrix}\right|.$$

Note $\det\begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\1 & 1 & 1\end{bmatrix} = \det\begin{bmatrix}a_1 & b_1-a_1 & c_1-a_1\\a_2 & b_2-a_2 & c_2-a_2\\1 & 0 & 0\end{bmatrix}$.

$= \det\begin{bmatrix}b_1-a_1 & c_1-a_1\\b_2-a_2 & c_2-a_2\end{bmatrix}$, so area of triangle $= \frac{1}{2}\left|\det\begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\1 & 1 & 1\end{bmatrix}\right|$.

7. Find the area of the following region:



this region can be divided into two triangles S_1 & S_2:

S_1 enclosed by $\begin{bmatrix}-7\\7\end{bmatrix}$. $\begin{bmatrix}-5\\6\end{bmatrix}$. $\begin{bmatrix}3\\-4\end{bmatrix}$. hence by Ex 4 above.

area of $S_1 = \frac{1}{2}\left|\det\begin{bmatrix}-7 & -5 & 3\\7 & 6 & -4\\1 & 1 & 1\end{bmatrix}\right| = \frac{1}{2}\left|\det\begin{bmatrix}-7 & 2 & 10\\7 & -1 & -11\\1 & 0 & 0\end{bmatrix}\right|$

$$= \frac{1}{2}\left|\det\begin{bmatrix}2 & 10\\-1 & -11\end{bmatrix}\right| = \frac{1}{2}\left|-22+10\right| = \frac{1}{2}\cdot 12 = 6.$$

S_2 enclosed by $\begin{bmatrix}-7\\7\end{bmatrix}$. $\begin{bmatrix}5\\5\end{bmatrix}$ & $\begin{bmatrix}3\\-4\end{bmatrix}$. hence area of $S_2 = \frac{1}{2}\left|\det\begin{bmatrix}-7 & 5 & 3\\7 & 5 & -4\\1 & 1 & 1\end{bmatrix}\right|$.

$$= \frac{1}{2}\left|\det\begin{bmatrix}-7 & 12 & 10\\7 & -2 & -11\\1 & 0 & 0\end{bmatrix}\right| = \frac{1}{2}\left|\det\begin{bmatrix}12 & 10\\-2 & -11\end{bmatrix}\right| = \frac{1}{2}\left|-132+20\right| = \frac{1}{2}\cdot 112 = 56.$$

$\Rightarrow$ area of the region $=$ area of S_1 $+$ area of $S_2 = 6 + 56 = 62$.

14. Find the 3-volume of the 3-parallelepiped defined by the vectors

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

By theorem 6.3.6. this volume $= \sqrt{\det(A^TA)}$, where $A = \begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix}$ so we have.

$$A^TA = \begin{bmatrix}1&0&0&0\\1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix} = \begin{bmatrix}1&1&1\\1&4&10\\1&10&30\end{bmatrix} \xrightarrow[\text{III}-\text{I}]{\text{II}-\text{I}} \begin{bmatrix}1&1&1\\0&3&9\\0&9&29\end{bmatrix} \xrightarrow{\text{III}-3\cdot\text{II}} \begin{bmatrix}1&1&1\\0&3&9\\0&0&2\end{bmatrix}.$$

$\Rightarrow \det(A^TA) = 1\cdot3\cdot2 = 6. \Rightarrow$ the 3-volume $= \sqrt{6}$.

29. the system is given by

$$\begin{bmatrix}-R_1 & R_1 & -(1-\alpha)\\ \alpha & (1-\alpha) & -(1-\alpha)^2\\ R_2 & -R_2 & \frac{-(1-\alpha)^2}{\alpha}\end{bmatrix}\begin{bmatrix}dx_1\\dy_1\\dp\end{bmatrix} = \begin{bmatrix}0\\0\\-R_2de_2\end{bmatrix}. \qquad R_1, R_2, D>0.\ \alpha\in(0,1).$$

using Cramer's rule, we have $dy_1 = \dfrac{\det\begin{bmatrix}-R_1 & 0 & -(1-\alpha)\\ \alpha & 0 & -(1-\alpha)^2\\ R_2 & -R_2de_2 & -(1-\alpha)^2/\alpha\end{bmatrix}}{D}$

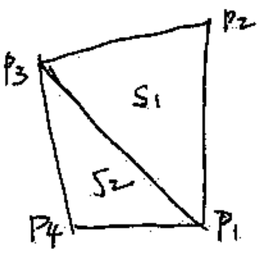
$$= \frac{R_2de_2\cdot\det\begin{bmatrix}-R_1 & -(1-\alpha)\\ \alpha & -(1-\alpha)^2\end{bmatrix}}{D} = \frac{(R_1(1-\alpha)^2+\alpha(1-\alpha))R_2de_2}{D} > 0$$

(since $R_1>0$, $\alpha>0$, $1-\alpha>0$, $R_2>0$, $de_2>0$ & $D>0$). $\Rightarrow dy_1$ is positive.

Similarly, $dp = \det\begin{bmatrix}-R_1 & R_1 & 0\\ \alpha & (1-\alpha) & 0\\ R_2 & -R_2 & -R_2de_2\end{bmatrix}/D = \dfrac{-R_2de_2\det\begin{bmatrix}-R_1 & R_1\\ \alpha & (1-\alpha)\end{bmatrix}}{D} = \dfrac{-R_2de_2\cdot(-R_1(1-\alpha)-R_1\alpha)}{D}$

$= \dfrac{-R_2de_2\cdot(-R_1)}{D} = R_1R_2de_2/D > 0$ (R_1, R_2, de_2, D all >0) $\Rightarrow dp$ is also positive.

47. Consider the following quadrilateral with vertices $P_i = (x_i, y_i)$ for $i=1,2,3,4$. Show that the area of this is $\frac{1}{2}\left(\det\begin{bmatrix}x_1&x_2\\y_1&y_2\end{bmatrix} + \det\begin{bmatrix}x_2&x_3\\y_2&y_3\end{bmatrix} + \det\begin{bmatrix}x_3&x_4\\y_3&y_4\end{bmatrix} + \det\begin{bmatrix}x_4&x_1\\y_4&y_1\end{bmatrix}\right)$



area = area of the triangle S_1 + area of triangle S_2.

using exercise 4. area of $S_1 = \frac{1}{2}\det\begin{bmatrix}x_1&x_2&x_3\\y_1&y_2&y_3\\1&1&1\end{bmatrix}$

area of $S_2 = \frac{1}{2}\det\begin{bmatrix}x_1&x_3&x_4\\y_1&y_3&y_4\\1&1&1\end{bmatrix}$.

Laplace expansion

$$\Rightarrow \text{area} = \frac{1}{2}\det\begin{bmatrix}x_1&x_2&x_3\\y_1&y_2&y_3\\1&1&1\end{bmatrix} + \frac{1}{2}\det\begin{bmatrix}x_1&x_3&x_4\\y_1&y_3&y_4\\1&1&1\end{bmatrix} = \frac{1}{2}\left[\det\begin{bmatrix}x_2&x_3\\y_2&y_3\end{bmatrix} - \cancel{\det\begin{bmatrix}x_1&x_3\\y_1&y_3\end{bmatrix}} + \det\begin{bmatrix}x_1&x_2\\y_1&y_2\end{bmatrix}\right.$$

$$\left. + \det\begin{bmatrix}x_3&x_4\\y_3&y_4\end{bmatrix} - \det\begin{bmatrix}x_1&x_4\\y_1&y_4\end{bmatrix} + \cancel{\det\begin{bmatrix}x_1&x_3\\y_1&y_3\end{bmatrix}}\right] = \frac{1}{2}\left[\det\begin{bmatrix}x_1&x_2\\y_1&y_2\end{bmatrix} + \det\begin{bmatrix}x_2&x_3\\y_2&y_3\end{bmatrix} + \det\begin{bmatrix}x_3&x_4\\y_3&y_4\end{bmatrix}\right.$$

$$\left. + \det\begin{bmatrix}x_4&x_1\\y_4&y_1\end{bmatrix}\right)$$ as claimed.