Lin.Alg. HW6
Solutions

Linear Algebra

§5.1 // 4) $u \cdot v = \|u\| \|v\| \cos\theta$

$$\begin{bmatrix}1\\1\end{bmatrix} \cdot \begin{bmatrix}7\\11\end{bmatrix} = (\sqrt{2})(\sqrt{170}) \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{18}{\sqrt{340}}\right) = 12.53°$$

6) $u \cdot v = \|u\| \|v\| \cos\theta$

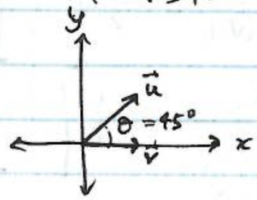
$$\begin{bmatrix}1\\-1\\2\\-2\end{bmatrix} \cdot \begin{bmatrix}2\\3\\4\\5\end{bmatrix} = (\sqrt{10})(\sqrt{54}) \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{2-3+8-10}{\sqrt{540}}\right) = 97.42°$$

10 11) a. $n=2$ $\quad u \cdot v = \|u\| \|v\| \cos\theta$

$$\begin{bmatrix}1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\end{bmatrix} = (\sqrt{2})(1) \cos\theta$$

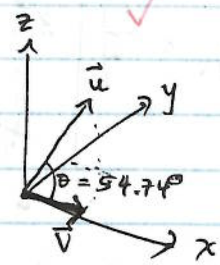
$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$



$n=3$ $\quad u \cdot v = \|u\| \|v\| \cos\theta$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = (\sqrt{3})(1) \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.74°$$



$n=4$ $\quad u \cdot v = \|u\| \|v\| \cos\theta$

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = (2)(1) \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°$$

b. $$\lim_{n\to\infty} \theta = \cos^{-1}\left(\frac{1}{\infty}\right) = \cos^{-1}(0) = 90°$$

26) $v_1 = \begin{bmatrix}2\\3\\6\end{bmatrix}$ $\quad v_2 = \begin{bmatrix}3\\-6\\2\end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix}49\\49\\49\end{bmatrix}$

$$u_1 = \frac{1}{\sqrt{4+9+36}} \begin{bmatrix}2\\3\\6\end{bmatrix} = \frac{1}{7}\begin{bmatrix}2\\3\\6\end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{9+36+4}} \begin{bmatrix}3\\-6\\2\end{bmatrix} = \frac{1}{7}\begin{bmatrix}3\\-6\\2\end{bmatrix}$$

$$v_2^{\perp} = v_2 - v_2^{\parallel} = v_2 - (u_1 \cdot v_2) u_1$$

$$= \begin{bmatrix}3\\-6\\2\end{bmatrix} - \left(\frac{1}{49}\right)(6-18+12)\begin{bmatrix}2\\3\\6\end{bmatrix}$$

$$= \begin{bmatrix}3\\6\\2\end{bmatrix}$$

$$\text{Proj}_V \vec{x} = (\vec{x} \cdot \vec{u}_1) u_1 + (\vec{x} \cdot \vec{u}_2) u_2$$

$$= (2+3+6)\begin{bmatrix}2\\3\\6\end{bmatrix} + (3-6+2)\begin{bmatrix}3\\-6\\2\end{bmatrix} = \begin{bmatrix}22\\33\\66\end{bmatrix} - \begin{bmatrix}3\\-6\\2\end{bmatrix} = \begin{bmatrix}19\\39\\64\end{bmatrix}$$

10 28) $v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $v_2 = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$ $v_3 = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$ $\vec{x} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

$u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$u_2 = \dfrac{v_2}{\|v_2\|} = \dfrac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$

$u_3 = \dfrac{v_3}{\|v_3\|} = \dfrac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

$v_1 \cdot v_2 = 0$

$v_2 \cdot v_3 = 0$

$v_3 \cdot v_1 = 0$

$\text{proj}_V \vec{x} = (u_1 \cdot x)u_1 + (u_2 \cdot x)u_2 + (u_3 \cdot x)u_3$

$= \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{4}\\\frac{1}{4}\\-\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$

$\text{Proj}_V \vec{x} = \frac{1}{4}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix}$ ✓

5.1 // 40) $\|\vec{v}_2\|^2 = v_2 \cdot v_2 = 9$

$\|v_2\| = 3$

41) $v_2 \cdot v_3 = \|v_2\|\|v_3\| \cos\theta$

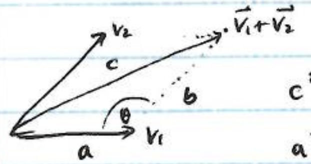
$\|v_3\|^2 = v_3 \cdot v_3 = 49$

$\|v_3\| = 7$

$20 = (3)(7)\cos\theta$

$\theta = \cos^{-1}\left(\frac{20}{21}\right) = 17.75°$

42)



$c^2 = a^2 + b^2 - 2ab\cos\theta$

$a^2 = \|v_1\| = v_1 \cdot v_1 = 3$

$b^2 = \|v_2\| = v_2 \cdot v_2 = 9$

$\theta = 180 - \theta_1$

$= 164.21°$

$v_1 \cdot v_2 = \|v_1\|\|v_2\|\cos\theta_1$

$5 = (\sqrt{3})(3)\cos\theta_1$

$\theta_1 = \cos^{-1}\left(\frac{5}{3\sqrt{3}}\right)$

$\|v_1 + v_2\|^2 = 3 + 9 - 2(\sqrt{3})(3)\cos(164.21°)$

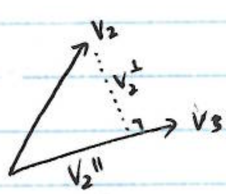
$= 22$

$\|v_1 + v_2\| = \sqrt{22}$

43) $\text{proj}_{v_2}(\vec{v}_1) = \dfrac{v_1 \cdot v_2}{v_2 \cdot v_2} v_2$

$= \dfrac{5}{9} v_2$

44)



$v_2 = v_2^\parallel + v_2^\perp$

$\text{proj}_{\vec{v}_3}(\vec{v}_2) = \dfrac{v_2 \cdot v_3}{v_3 \cdot v_3} v_3$

$= \dfrac{20}{49} v_3$

$\boxed{v_2^\perp = v_2 - \frac{20}{49} v_3}$

45) $u_3 = \frac{1}{\|v_3\|} v_3 = \frac{1}{7} v_3$ $\quad u_2^{\perp} = \frac{v_2^{\perp}}{\|v_2^{\perp}\|}$

$V = \text{span}(u_2, u_3)$ $\quad u_2 = \frac{7}{\sqrt{41}} v_2^{\perp}$

$$\|v_2^{\perp}\|^2 = v_2^{\perp} \cdot v_2^{\perp}$$
$$= \left(v_2 - \frac{20}{49} v_3\right) \cdot \left(v_2 - \frac{20}{49} v_3\right)$$
$$= v_2 \cdot v_2 + \frac{20^2}{49^2} v_3 \cdot v_3 + 2\left(-\frac{20}{49}\right) v_2 \cdot v_3$$
$$= 9 + \frac{20^2}{49^{\cancel{2}}} \cancel{49} - \frac{40}{49}(20)$$
$$= \frac{41}{49} \qquad \|v_2^{\perp}\| = \frac{\sqrt{41}}{7}$$

$$\text{proj}_V(\vec{v_1}) = (u_2 \cdot v_1) u_2 + (u_3 \cdot v_1) u_3$$
$$= \left[\frac{7}{\sqrt{41}}\left(v_2 - \frac{20}{49} v_3\right) \cdot v_1\right]\left(\frac{7}{\sqrt{41}}\left(v_2 - \frac{20}{49} v_3\right)\right) + \frac{11 v_3}{49}$$
$$= \frac{49}{41}\left[(v_1 \cdot v_2) - \frac{20}{49}(v_1 \cdot v_3)\right]\left(v_2 - \frac{20}{49} v_3\right) + \frac{11}{49} v_3$$
$$= \frac{49}{41}\left[5 - \frac{20}{49}(11)\right]\left(v_2 - \frac{20}{49} v_3\right) + \frac{11}{49} v_3$$
$$= \frac{\cdot 25}{41}(v_2) - \frac{25 \cdot 20}{49 \cdot 41} v_3 + \frac{11}{49} v_3$$
$$= \frac{25}{41} v_2 - \frac{1}{41} v_3$$

10 46) $u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} v_1$ $\quad v_2^{\perp} = v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1 = v_2 - \frac{5}{3} v_1$

$u_2 = \frac{v_2^{\perp}}{\|v_2^{\perp}\|} = \frac{\sqrt{3}}{\sqrt{2}}\left(v_2 - \frac{5}{3} v_1\right)$

$$\|v_2^{\perp}\|^2 = v_2^{\perp} \cdot v_2^{\perp}$$
$$= \left(v_2 - \frac{5}{3} v_1\right) \cdot \left(v_2 - \frac{5}{3} v_1\right)$$
$$= (v_2 \cdot v_2) + \frac{25}{9} v_1 \cdot v_1 - \frac{2(5)}{3} v_1 \cdot v_2$$
$$= 9 + \frac{25(\cancel{3})}{\cancel{9}\,3} - \frac{10}{3}(5) = \frac{2}{3}$$

$V = \text{span}(u_1, u_2)$

$$\text{proj}_V(\vec{v_3}) = (u_1 \cdot v_3) u_1 + (u_2 \cdot v_3) u_2$$
$$= \left[\frac{1}{\sqrt{3}}(v_1 \cdot v_3)\right]\left[\frac{1}{\sqrt{3}} v_1\right] + \left[\frac{\sqrt{3}}{\sqrt{2}}\left(v_2 - \frac{5}{3} v_1\right) \cdot v_3\right]\left[\frac{\sqrt{3}}{\sqrt{2}}\left(v_2 - \frac{5}{3} v_1\right)\right]$$
$$= \frac{11}{3} v_1 + \frac{3}{2}\left(v_2 \cdot v_3 - \frac{5}{3} v_1 \cdot v_3\right)\left(v_2 - \frac{5}{3} v_1\right)$$
$$= \frac{11}{3} v_1 + \frac{3}{2}\left(20 - \frac{55}{3}\right)\left(v_2 - \frac{5}{3} v_1\right) = \frac{11}{3} v_1 + \frac{5}{2}\left(v_2 - \frac{5}{3} v_1\right)$$
$$= \frac{5}{2} v_2 - \frac{1}{2} v_1 \checkmark$$

§5.2 // 2) $v_1 = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$ $v_1 \cdot v_2 = 0$

So just normalize v_1 and v_2.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{36+9+4}} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{7} \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

6) $v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2^{\perp} = v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1 = v_2 - \frac{6}{4} v_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{v_2^{\perp}}{\|v_2^{\perp}\|} = \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_3 = \frac{v_3^{\perp}}{\|v_3^{\perp}\|} = \frac{1}{7} \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_3^{\perp} = v_3 - (u_1 \cdot v_3) u_1 - (u_2 \cdot v_3) u_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

10 14) $v_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{10} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} \checkmark$$

$$v_2^{\perp} = v_2 - (u_1 \cdot v_2) u_1 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - \frac{100}{100} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{v_2^{\perp}}{\|v_2^{\perp}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

$$v_3^{\perp} = v_3 - (u_1 \cdot v_3) u_1 - (u_2 \cdot v_3) u_2 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} - 0 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{100}{100} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$u_3 = \frac{v_3^{\perp}}{\|v_3^{\perp}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \checkmark$$

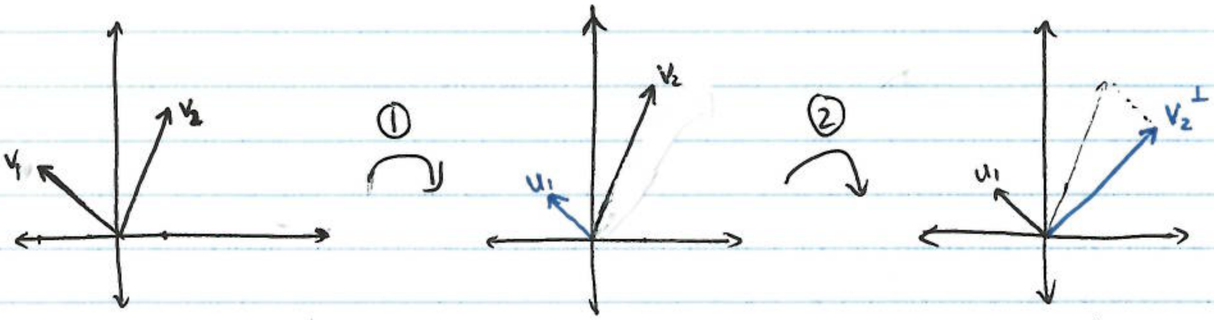
29) $\vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

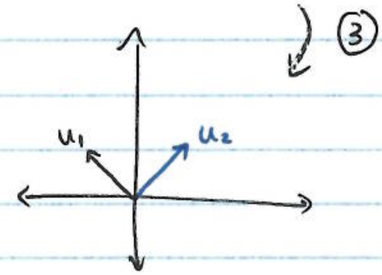
① $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

② $v_2^{\perp} = v_2 - \dfrac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1$

$= \begin{bmatrix} 1 \\ 7 \end{bmatrix} - \dfrac{25}{25}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} \\ 3 \end{bmatrix}$

③ $u_2 = \dfrac{v_2^{\perp}}{\|v_2^{\perp}\|} = \dfrac{1}{\sqrt{13}}\begin{bmatrix} \\ 3 \end{bmatrix}$





10 34) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$\ker(A) = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

$v_2^{\perp} = v_2 - \dfrac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1$

$= \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} - \dfrac{8}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$

$u_2 = \dfrac{v_2^{\perp}}{\|v_2^{\perp}\|} = \dfrac{\cancel{3}}{\cancel{3}\sqrt{30}}\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$

$u_1 = \dfrac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ ✓, $u_2 = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$ ✓

$\frac{60}{60}$