4) $\vec{v}_1$ and $\vec{v}_2$ aren't scalars of each other $\Rightarrow$ lin. indep.
Thus, because we have two lin. indep. vectors in $\mathbb{R}^2$, they form a basis for $\mathbb{R}^2$
$\Rightarrow \vec{x}$ is in their span.

$$S=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow S^{-1}=\frac{1}{0-1}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow [\vec{x}]_\beta=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix}=\begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

6) If we want c_1, c_2 s.t. $\vec{x}=c_1\vec{v}_1+c_2\vec{v}_2$, we see from the 2^{nd} + 3^{rd} coordinates that $c_1=3, c_2=4$, but that would result in a 1^{st} coordinate of 11, so this can't work. Thus, $\vec{x}$ isn't in the span of $\vec{v}_1$ and $\vec{v}_2$.

16) We can see that to satisfy $\vec{x}=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3$, we'll need $c_1=3$, and then $c_2=4$, then $c_3=6$ (just by going through the standard basis coordinates of $\vec{x}$). The existence of these scalars then tells us that $\vec{x}\in \text{span}(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ and that $[\vec{x}]_\beta=\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

18) If $\vec{x}=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3$, we can see that the 1^{st}, 3^{rd}, and 4^{th} standard basis coordinates of $\vec{x}$ will be given by c_1, c_2, and c_3, respectively $\Rightarrow c_1=5$, $c_2=3$, $c_3=2$. But then the lin. comb. gives a 2^{nd} coordinate of $6\neq 4$, so $\vec{x}\notin \text{span}(\vec{v}_1,\vec{v}_2,\vec{v}_3)$.

26) First, we note that $S=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \Rightarrow S^{-1}=\frac{1}{1-2}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

Now, $B=\begin{bmatrix} | & | \\ [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta \\ | & | \end{bmatrix}=\begin{bmatrix} | & | \\ [A\vec{v}_1]_\beta & [A\vec{v}_2]_\beta \\ | & | \end{bmatrix}=\begin{bmatrix} | & | \\ S^{-1}A\vec{v}_1 & S^{-1}A\vec{v}_2 \\ | & | \end{bmatrix}$.

$$S^{-1}A=\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}=\begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix} \Rightarrow \begin{cases} S^{-1}A\vec{v}_1=\begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \\ S^{-1}A\vec{v}_2=\begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 4 \\ -3 \end{bmatrix} \end{cases}$$

$$\Rightarrow B=\begin{bmatrix} 6 & 4 \\ -4 & -3 \end{bmatrix}.$$

28) Similarly to the last problem, we have $S=\begin{bmatrix}2&1&0\\2&-1&1\\1&0&-2\end{bmatrix} \Rightarrow S^{-1}=\frac{1}{9}\begin{bmatrix}2&2&1\\5&-4&-2\\1&1&-4\end{bmatrix}$

$\Rightarrow S^{-1}A=\frac{1}{9}\begin{bmatrix}2&2&1\\5&-4&-2\\1&1&-4\end{bmatrix}\begin{bmatrix}5&-4&-2\\-4&5&-2\\-2&-2&8\end{bmatrix}=\frac{1}{9}\begin{bmatrix}0&0&0\\45&-36&-18\\9&9&-36\end{bmatrix}=\begin{bmatrix}0&0&0\\5&-4&-2\\1&1&-4\end{bmatrix}$

$\Rightarrow S^{-1}A\vec{v}_1=\begin{bmatrix}0&0&0\\5&-4&-2\\1&1&-4\end{bmatrix}\begin{bmatrix}2\\2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, $S^{-1}A\vec{v}_2=\begin{bmatrix}0&0&0\\5&-4&-2\\1&1&-4\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\9\\0\end{bmatrix}$, $S^{-1}A\vec{v}_3=\begin{bmatrix}0&0&0\\5&-4&-2\\1&1&-4\end{bmatrix}\begin{bmatrix}0\\1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\9\end{bmatrix}$

$\Rightarrow B=\begin{bmatrix}0&0&0\\0&9&0\\0&0&9\end{bmatrix}$.

30) Similarly to the last two problems, we have $S=\begin{bmatrix}1&0&1\\1&1&2\\1&2&4\end{bmatrix} \Rightarrow S^{-1}=\begin{bmatrix}0&2&-1\\-2&3&-1\\1&-2&1\end{bmatrix}$

$\Rightarrow S^{-1}A=\begin{bmatrix}0&2&-1\\-2&3&-1\\1&-2&1\end{bmatrix}\begin{bmatrix}0&2&-1\\2&-1&0\\4&-4&1\end{bmatrix}=\begin{bmatrix}0&2&-1\\2&-3&1\\0&0&0\end{bmatrix}$

$\Rightarrow S^{-1}A\vec{v}_1=\begin{bmatrix}0&2&-1\\2&-3&1\\0&0&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $S^{-1}A\vec{v}_2=\begin{bmatrix}0&2&-1\\2&-3&1\\0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$, $S^{-1}A\vec{v}_3=\begin{bmatrix}0&2&-1\\2&-3&1\\0&0&0\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix}=\begin{bmatrix}8\\0\\0\end{bmatrix}$

$\Rightarrow B=\begin{bmatrix}1&0&8\\0&-1&0\\0&0&0\end{bmatrix}$.

38) B diagonal $\Rightarrow$ T transforms $\vec{v}_1$ into a scalar multiple of itself, and the same for $\vec{v}_2$. For a reflection, this will only happen for vectors parallel or perpendicular to the line of reflection, so we choose one vector of each type for linear independence (they must form a basis): $\vec{v}_1=\begin{bmatrix}2\\3\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}-3\\2\end{bmatrix}$. These correspond to $B=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ ($\vec{v}_1$ remains the same under T while $\vec{v}_2$ is multiplied by -1).

39) The situation is the same as the last problem, except now we need two lin. indep. vectors perpendicular to the line of reflection. By inspection, we see that $\vec{v}_2=\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\vec{v}_3=\begin{bmatrix}-3\\0\\1\end{bmatrix}$ do the trick. So, setting $\vec{v}_1=\begin{bmatrix}1\\2\\3\end{bmatrix}$, we get a basis for which $B=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$.

56) We have that $\begin{bmatrix}3\\5\end{bmatrix} = S\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\3\end{bmatrix} = S\begin{bmatrix}3\\4\end{bmatrix}$ for some 2×2 matrix S.
So, setting $S = \begin{bmatrix}s_1 & s_2\\ s_3 & s_4\end{bmatrix}$, we get that:

$$\begin{cases}\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}s_1 & s_2\\ s_3 & s_4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}, \\ \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}s_1 & s_2\\ s_3 & s_4\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}\end{cases} \Rightarrow \begin{cases} s_1 + 2s_2 = 3, \\ s_3 + 2s_4 = 5, \\ 3s_1 + 4s_2 = 2, \\ 3s_3 + 4s_4 = 3 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 5 \\ 3 & 4 & 0 & 0 & 2 \\ 0 & 0 & 3 & 4 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 7/2 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

$$\Rightarrow S = \begin{bmatrix}-4 & 7/2\\ -7 & 6\end{bmatrix} \Rightarrow \beta = \left(\begin{bmatrix}-4\\-7\end{bmatrix}, \begin{bmatrix}7/2\\6\end{bmatrix}\right).$$

60) We want to see if $\exists$ S s.t. $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}S = S\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. Setting $S = \begin{bmatrix}s_1 & s_2\\ s_3 & s_4\end{bmatrix}$, we get that $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}s_1 & s_2\\ s_3 & s_4\end{bmatrix} = \begin{bmatrix}s_1 & s_2\\ s_3 & s_4\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}s_1 & s_2\\ -s_3 & -s_4\end{bmatrix} = \begin{bmatrix}s_2 & s_1\\ s_4 & s_3\end{bmatrix}$. This is satisfied whenever $s_1 = s_2$ and $s_3 = -s_4$, so we can choose $S = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ to satisfy $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}S = S\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. Because such an S exists, $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ are similar.

67) We have $S = \begin{bmatrix}1 & a\\ 0 & c\end{bmatrix} \Rightarrow S^{-1} = \frac{1}{c-0}\begin{bmatrix}c & -a\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -a/c\\ 0 & 1/c\end{bmatrix}$

$$\Rightarrow S^{-1}A = \begin{bmatrix}1 & -a/c\\ 0 & 1/c\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a-a & b-ad/c\\ 1 & d/c\end{bmatrix} = \begin{bmatrix}0 & b-ad/c\\ 1 & d/c\end{bmatrix}$$

$$\Rightarrow S^{-1}A\vec{v}_1 = \begin{bmatrix}0 & b-ad/c\\ 1 & d/c\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}, \quad S^{-1}A\vec{v}_2 = \begin{bmatrix}0 & b-ad/c\\ 1 & d/c\end{bmatrix}\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}bc-ad\\ a+d\end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix}0 & bc-ad\\ 1 & a+d\end{bmatrix}.$$