2.3] 7) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+0-2 & 2+0-1 & 3+0-3 \\ 0+3+2 & 0+2+1 & 0+1+3 \\ 1-3-4 & 2-2-2 & 3-1-6 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$

8) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+da \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)I$

29) a. Multiplying two linear transformations' matrices corresponds to composing the two transformations. Thus, $T_1(\vec{x}) = D_\alpha D_\beta \vec{x}$ rotates $\vec{x}$ first by $\alpha°$ and second by $\beta°$, both times counterclockwise, and $T_2(\vec{x}) = D_\beta D_\alpha \vec{x}$ is the opposite. In either case, we're rotating counterclockwise by an angle of $(\alpha+\beta)°$, so T_1 and T_2 are the same linear transformation.

b. $D_\alpha D_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$

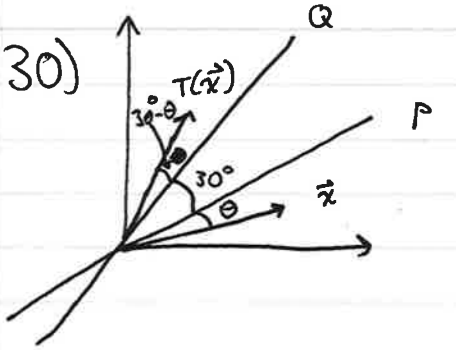
$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = D_{\alpha+\beta}$, and

$D_\beta D_\alpha = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ \cos\beta\sin\alpha + \sin\beta\cos\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = D_{\alpha+\beta}.$

Thus, just as we expected, $D_\alpha D_\beta = D_\beta D_\alpha$ and their product corresponds to the matrix that represents a counterclockwise rotation by $\alpha+\beta$.

30)



a. Reflections don't change vectors' lengths, so $\|\vec{x}\| = \|T(\vec{x})\|$.
If the angle between $\vec{x}$ and P is θ as in the diagram, then $\vec{x}$ rotates counterclockwise by an angle of 2θ when reflected over P and then makes an angle of $30°-\theta$ with Q. So, when we then reflect over Q, the resulting angle between $T(\vec{x})$ and Q is $30°-\theta$. As can be seen in the diagram, this will correspond to a total counterclockwise rotation of $\theta + 30° + 30° - \theta = 60°$, meaning $\vec{x}$ and $T(\vec{x})$ enclose an angle of $60°$.

b. Because the rotation in (a) doesn't depend on θ, T is a counterclockwise rotation by $60°$.

c. Using the formula for a rotation matrix and plugging in $\theta = 60°$, we have that T's corresponding matrix is $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

d. L represents the inverse transformation of T, as reflecting a vector back over Q then back over P will exactly undo T's work on a vector. Thus, L corresponds to a clockwise rotation of $60°$, meaning its matrix is $\begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$.

38) $A^2 = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} - \frac{3}{4} & \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & -\frac{3}{4} + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}.$

$A^3 = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \frac{3}{4} & -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} & \frac{3}{4} + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

$A^4 = IA = A.$

Thus, the pattern repeats every 3 integers, meaning that because $1001 = 3(111)+2$, A^{1001} looks like $A^2 \Rightarrow A^{1001} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}.$

A corresponds to a counterclockwise rotation by 120°, so it makes sense that three applications of A is equivalent to doing nothing (rotate by 360° = rotate by 0°).

2.4] 1) $\overset{A}{\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 5 & 8 & 0 & 1 \end{array}\right]} \div 2 \rightarrow \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 5 & 8 & 0 & 1 \end{array}\right] -5 \cdot R1 \rightarrow \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1/2 & -5/2 & 1 \end{array}\right] \times 2$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right] -\frac{3}{2} \cdot R2 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 8 & -3 \\ 0 & 1 & -5 & 2 \end{array}\right].$ Thus, A is invertible and $A^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}.$

6) $\overset{A}{\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]} \begin{array}{l} \\ -R1 \\ -R1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \times(-\frac{1}{2}) \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} -2R2 \\ \\ -R2 \end{array}$

$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right] \begin{array}{l} -R3 \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right].$

Thus, A is invertible and $A^{-1} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}.$

8) $$\overset{A}{\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right]} \begin{array}{l} \\ -R1 \\ -R1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -R2 \\ \\ -2R2 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} +R3 \\ -2R3 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right].$$ Thus, A is invertible and $A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

14) $$\overset{A}{\left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 5 & 4 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]} \begin{array}{l} \\ \times(-1) \\ -2R1 \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -R2 \\ \\ \\ -3R2 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right] \begin{array}{l} -2R3 \\ \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 7 & 5 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right] \begin{array}{l} -7R4 \\ \\ +2R4 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5 & -20 & -2 & -7 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right].$$ Thus, A is invertible and $A^{-1} = \begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$.

28) T represents left-multiplication by $A = \begin{bmatrix} 22 & 13 & 8 & 3 \\ -16 & -3 & -2 & -2 \\ 8 & 9 & 7 & 2 \\ 5 & 4 & 3 & 1 \end{bmatrix}$, so we need only find A^{-1}! Using a computer, we get that

$$A^{-1} = \begin{bmatrix} 1 & -2 & 9 & -25 \\ -2 & 5 & -22 & 60 \\ 4 & -9 & 41 & -112 \\ -9 & 17 & -80 & 222 \end{bmatrix} \Rightarrow T^{-1}(\vec{x}) = A^{-1}\vec{x}.$$

3.1] 6) We wish to find all vectors $\vec{x}$ in $\mathbb{R}^3$ s.t. $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \begin{array}{c} \\ -R1 \end{array} \rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{c} -R2 \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = x_3, \\ x_2 = -2x_3 \end{cases}$$

$\Rightarrow x_3$ is a free variable, so in general we have that solutions take the form $t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

$$\Rightarrow \ker(A) = \left\{ t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

15) We begin by putting A in RREF:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{array}{c} \\ -R1 \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{array}{c} -R2 \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Columns 1 and 2 are the only ones with pivot variables, so $\text{im}(A)$ is spanned by its first two columns, and both columns are necessary (A's rank is 2, so we need at least 2 vectors to span $\text{im}(A)$. Thus,

$$\text{im}(A) = \left\{ t_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\}.$$

24) If we're projecting everything onto this plane P, then $\mathrm{im}(A)$ is a subset of the plane. Additionally, we'll have that $A\vec{x}=\vec{x}\ \forall\, \vec{x}\in P$, as any vector on the plane will just be projected to itself, so because $P\subseteq \mathbb{R}^3$ we'll have that $P\subseteq \mathrm{im}(A)$. Thus, $\mathrm{im}(A)=P$, or in other words the image of an orthogonal projection onto a plane is the plane itself.

The kernel of this transformation will be composed of all vectors that are sent to the origin when projected orthogonally onto P. So, the kernel will naturally be a line through the origin along a normal vector to P. In this case, such a vector is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ (we can get this by extracting the coefficients in P's equation) $\Rightarrow \ker(A)=\left\{t\begin{bmatrix}1\\2\\3\end{bmatrix} : t\in\mathbb{R}\right\}$.

34) As seen in the previous problem, if we consider the linear transformation T w/ corresponding matrix A that orthogonally projects $\mathbb{R}^3$ onto the plane through the origin with normal vector $\begin{bmatrix}-1\\1\\2\end{bmatrix}$, then T's kernel will be precisely the line spanned by $\begin{bmatrix}-1\\1\\2\end{bmatrix}$. So, the linear transformation in question is the orthogonal projection of $\mathbb{R}^3$ onto the plane $-x+y+2z=0$. To find this transformation's matrix, we go through the following process, where the $\parallel$ and $\perp$ superscripts refer to $\begin{bmatrix}-1\\1\\2\end{bmatrix}$:

$$\vec{e}_1^{\,\parallel}=\left(\frac{\vec{e}_1\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix}}{\begin{bmatrix}-1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix}}\right)\begin{bmatrix}-1\\1\\2\end{bmatrix}=\frac{-1}{6}\begin{bmatrix}-1\\1\\2\end{bmatrix}=\begin{bmatrix}1/6\\-1/6\\-1/3\end{bmatrix}\Rightarrow \vec{e}_1^{\,\perp}=\vec{e}_1-\vec{e}_1^{\,\parallel}=\begin{bmatrix}1\\0\\0\end{bmatrix}-\begin{bmatrix}1/6\\-1/6\\-1/3\end{bmatrix}=\begin{bmatrix}5/6\\1/6\\1/3\end{bmatrix};$$

$$\vec{e}_2^{\,\parallel}=\left(\frac{\vec{e}_2\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix}}{\begin{bmatrix}-1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix}}\right)\begin{bmatrix}-1\\1\\2\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-1\\1\\2\end{bmatrix}=\begin{bmatrix}-1/6\\1/6\\1/3\end{bmatrix}\Rightarrow \vec{e}_2^{\,\perp}=\vec{e}_2-\vec{e}_2^{\,\parallel}=\begin{bmatrix}0\\1\\0\end{bmatrix}-\begin{bmatrix}-1/6\\1/6\\1/3\end{bmatrix}=\begin{bmatrix}1/6\\5/6\\-1/3\end{bmatrix};$$

$$\vec{e}_3^{\,\parallel}=\left(\frac{\vec{e}_3\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix}}{\begin{bmatrix}-1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix}}\right)\begin{bmatrix}-1\\1\\2\end{bmatrix}=\frac{2}{6}\begin{bmatrix}-1\\1\\2\end{bmatrix}=\begin{bmatrix}-1/3\\1/3\\2/3\end{bmatrix}\Rightarrow \vec{e}_3^{\,\perp}=\vec{e}_3-\vec{e}_3^{\,\parallel}=\begin{bmatrix}0\\0\\1\end{bmatrix}-\begin{bmatrix}1/3\\-1/3\\1/3\end{bmatrix}.$$

Thus, $A=\begin{bmatrix}| & | & |\\ \vec{e}_1^{\,\perp} & \vec{e}_2^{\,\perp} & \vec{e}_3^{\,\perp}\\ | & | & |\end{bmatrix}=\begin{bmatrix}5/6 & 1/6 & 1/3\\ 1/6 & 5/6 & -1/3\\ 1/3 & -1/3 & 1/3\end{bmatrix}\Rightarrow T(\vec{x})=\begin{bmatrix}5/6 & 1/6 & 1/3\\ 1/6 & 5/6 & -1/3\\ 1/3 & -1/3 & 1/3\end{bmatrix}\vec{x}$.

42) This system will be consistent ~~all~~ precisely when the RHS of the bottom two rows of the row-reduced matrix the book gives us is 0 for each row. Thus, we need $\begin{cases} y_1 - 3y_3 + 2y_4 = 0 \\ y_2 - 2y_3 + y_4 = 0. \end{cases}$ The set of vectors $\vec{y}$ that satisfy this system is precisely the kernel of $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \Rightarrow im(A) = ker\left(\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}\right).$